

Vector Energy and Large Deviation

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Abstract

For d nonpolar compact sets $K_1, \dots, K_d \subset \mathbb{C}$, admissible weights Q_1, \dots, Q_d and a positive semidefinite interaction matrix $C = (c_{i,j})_{i,j=1,\dots,d}$ with no zero column, we define natural discretizations of the weighted energy

$$E_Q(\mu) := \sum_{i,j=1}^d c_{i,j} I(\mu_i, \mu_j) + 2 \sum_{j=1}^d \int_{K_j} Q_j d\mu_j$$

of a d -tuple of positive measures $\mu = (\mu_1, \dots, \mu_d) \in \mathcal{M}_r(K)$ where μ_j is supported in K_j and has mass r_j . We have an L^∞ -type discretization $W(\mu)$ and an L^2 -type discretization $J(\mu)$ defined using a fixed measure $\nu = (\nu_1, \dots, \nu_d)$. This leads to a large deviation principle for a canonical sequence $\{\sigma_k\}$ of probability measures on $\mathcal{M}_r(K)$ if ν is a strong Bernstein-Markov measure.

1 Introduction and main results

We prove a large deviations principle (LDP) which applies to the normalized counting measure of a random point in many multiple orthogonal polynomial ensembles, including Angelesco and certain Nikishin ensembles with compact supports. Our starting point is a very general vector energy setting first introduced in [12], [18] and further studied in [3], [15] and [13] associated to d compact sets $K_1, \dots, K_d \subset \mathbb{C}$, admissible weights Q_1, \dots, Q_d , and a positive semidefinite interaction matrix $C = (c_{i,j})_{i,j=1,\dots,d}$. We then define energy discretizations giving rise to the appropriate configuration space of points on the d -tuple of sets K_1, \dots, K_d .

Multiple orthogonal polynomials (MOPs) are a generalization of orthogonal polynomials in which the orthogonality is distributed among a number of orthogonality weights. They have been studied in connection with problems in analytic number theory, approximation theory and from the point of view of new special functions. In recent years MOPs have appeared in probability theory and certain models in mathematical physics coming from random matrices as MOPs can naturally give rise to ensembles of probability measures. This was first observed by Bleher and Kuijlaars [5] in the study of random matrix models with external source. Moreover, in the Gaussian case, the external source model is

equivalent to a model involving non-intersecting Brownian motion. An excellent account of the recent developments in the application of MOPs with extensive references can be found in [16] or [17]. Generally the MOPs have been studied using Riemann-Hilbert methods.

In this paper we are primarily concerned with the almost sure convergence of a random point in an ensemble to an equilibrium measure (Corollary 4.16) and a large deviation principle (Theorem 7.1). We begin with the discretization of a general vector energy. Two important special cases of the general ensembles we study in this paper are the Angelesco MOP case with interaction matrix $C = (c_{i,j})_{i,j=1,\dots,d}$ where $c_{i,i} = 1$ and $c_{i,j} = 1/2$ for $i \neq j$ and the Nikishin MOP case with interaction matrix given by $c_{i,i} = 1$, $c_{i,j} = -1/2$ if $|i-j| = 1$, $c_{i,j} = 0$ otherwise. These ensembles commonly arise from models in mathematical physics and they have a natural discretization – the points represent eigenvalues of matrices or positions of particles (see [16], [17]). In addition, β ensembles of random matrices correspond to a 1×1 interaction matrix consisting of a positive real number β and hence they may also be considered as a special case of the ensembles considered here.

In the case of disjoint compact intervals of the real line, a LDP for Angelesco ensembles was established in [6] using potential theory and in [15] where an extension of the method of Ben Arous-Guionnet [4, 2] was used. Recently, a LDP has also been obtained for the spectral measures of a non-centered Wishart matrix model, whose eigenvalue distribution can be described as a Nikishin ensemble in the presence of an external field on \mathbb{R}_+ and a constraint on \mathbb{R}_- , see [14]. In this paper we use potential theory and polynomial inequalities to establish our results, valid for *nonpolar compacta in \mathbb{C}* . We first prove the almost sure convergence of a random point to the equilibrium measure and subsequently establish the LDP. This method shows (Remark 7.2) that the rate function in the LDP is independent of the measure used to define L^2 norms as long as the measure satisfies a general condition, a *strong Bernstein-Markov property*.

The outline of the paper is as follows. In the next section we describe the vector energy minimization problems in the weighted and unweighted case. For clarity of exposition, we assume our compact sets are disjoint until section 8. The main idea is to give a discrete version of these energies $E(\mu)$ and $E_Q(\mu)$ following the ideas in [6]. These discretizations are L^∞ approximations and in order to develop the appropriate LDP, we need to introduce L^2 versions. This leads to notions of a (weighted) Bernstein-Markov property for vector measures which is the content of section 4. The utilization of a measure satisfying a strong (rational) Bernstein-Markov property is crucial for our approach to the LDP. To handle the case where some coefficients $c_{i,j}$ of C are negative we need to extend the notion of Bernstein-Markov property from polynomials to rational functions. In Theorem 4.5 we show that any nonpolar compact set in \mathbb{C} admits a measure satisfying a strong rational Bernstein-Markov property. Here we need to appeal to a result from [9] in \mathbb{C}^n for $n > 1$.

In particular, Proposition 4.14 shows that the asymptotics of a sequence of (weighted) L^2 “free energies” are the same as their L^∞ counterparts. To overcome a technical issue in the proof of our LDP in section 7, we must consider a non-admissible weighted problem, which shows up even in the scalar setting. We deal with this in section 5 using an approximation scheme with the aid of a deep result of Ancona [1]. We define our L^2 and L^∞ vector energy functionals J and W in section 6 culminating in the statement and proof of

our LDP in section 7. Section 8 indicates cases where our results remain valid, including an LDP, for possibly intersecting sets K_1, \dots, K_d .

2 Vector equilibrium problems

We begin with some potential-theoretic preliminaries in the scalar setting; i.e., associated to a single compact set. Let Q be an *admissible* weight on a nonpolar compact set $K \subset \mathbb{C}$. This means Q is lowersemicontinuous and finite on a set of positive logarithmic capacity; i.e., $\text{cap}(\{z \in K : Q(z) < +\infty\}) > 0$. The usual weighted energy minimization problem is:

$$\inf_{\mu \in \mathcal{M}(K)} \left(I(\mu) + 2 \int_K Q d\mu \right)$$

where $\mathcal{M}(K)$ denotes the probability measures on K and $I(\mu)$ is the standard logarithmic energy:

$$I(\mu) = \int_K \int_K \log \frac{1}{|z - t|} d\mu(z) d\mu(t) > -\infty.$$

We consider the slightly more general case where we minimize over $\mathcal{M}_r(K)$, the positive measures on K of total mass $r > 0$. We always have existence and uniqueness of a weighted energy minimizing measure $\mu^{K,Q}$. We write μ^K in the unweighted case ($Q \equiv 0$). Recalling that the logarithmic potential function of a measure μ is defined by

$$U^\mu(z) := \int \log \frac{1}{|z - t|} d\mu(t),$$

we say K is *regular* if U^{μ^K} is continuous. In the weighted case, there exists a constant F such that the logarithmic potential $U := U^{\mu^{K,Q}}$ satisfies

$$\begin{aligned} U(z) + Q &\geq F, \quad \text{q.e. } z \in K, \\ U(z) + Q &\leq F, \quad \forall z \in \text{supp}(\mu^{K,Q}) \end{aligned}$$

(“q.e.” means off of a polar set). Indeed, one can, in analogy with the case $r = 1$, define a weighted extremal function

$$V_Q(z) = \sup\{g(z) : g \in \mathcal{L}_r, g \leq Q \text{ on } K\},$$

and V_Q^* , its uppersemicontinuous regularization, where \mathcal{L}_r denotes the class of subharmonic functions in \mathbb{C} of growth at most $r \log |z|$ as $|z| \rightarrow \infty$. Then

$$V_Q^* = -U + F.$$

Let us now consider the vector case, where a d -tuple of nonpolar compact sets $K = (K_1, \dots, K_d)$ and a d -tuple of admissible weights $Q = (Q_1, \dots, Q_d)$ with Q_i defined on K_i , $i = 1, \dots, d$, are given, along with a symmetric positive semidefinite interaction matrix

$$C := (c_{i,j})_{i,j=1}^d,$$

with no zero columns (or rows). Throughout, until section 8, we assume that the sets K_i , $i = 1, \dots, d$, are *pairwise disjoint*. The unweighted energy of a d -tuple of measures $\mu = (\mu_1, \dots, \mu_d)$ is defined as

$$E(\mu) := \sum_{i,j=1}^d c_{i,j} I(\mu_i, \mu_j),$$

where $I(\mu_i, \mu_j)$ is the mutual energy:

$$I(\mu_i, \mu_j) = \int \int \log \frac{1}{|z - t|} d\mu_i(z) d\mu_j(t).$$

Note that, with the above assumptions, $I(\mu_i, \mu_j) \in (-\infty, \infty)$ if $i \neq j$. The weighted energy of $\mu = (\mu_1, \dots, \mu_d)$ is defined as

$$E_Q(\mu) := E(\mu) + 2 \sum_{i=1}^d \int Q_i d\mu_i.$$

We fix $r_1, \dots, r_d > 0$ and from now on we set

$$\mathcal{M}_r(K) := \{\mu = (\mu_1, \dots, \mu_d), \quad \mu_i \in \mathcal{M}_{r_i}(K_i), i = 1, \dots, d\}.$$

We equip $\mathcal{M}_r(K)$ with the (component-wise) weak-* topology. If we need to keep track of the underlying interaction matrix C , we write a superscript C ; e.g., E^C and E_Q^C . Note since $C \geq 0$ and since $-\log$ and Q are lowersemicontinuous functions, we have E and E_Q are lowersemicontinuous functionals on $\mathcal{M}_r(K)$ (see [18, Chapter 5, Proposition 4.1] and [3, Proposition 2.10] where the K_i may intersect).

From Theorem 1.8 of [3], it is known in the unweighted case there exists a unique minimizing d -tuple of measures for the energy E over $\mu \in \mathcal{M}_r(K)$ (for a positive definite C , the result is also proven in [18, Chapter 5]). We write this measure as $\mu^K = (\mu_1^K, \dots, \mu_d^K)$ and $E(\mu^K) = E^*$; in the weighted case there exists a unique minimizing tuple of measures for the energy E_Q and we write this measure as $\mu^{K,Q} = (\mu_1^{K,Q}, \dots, \mu_d^{K,Q})$ and $E_Q(\mu^{K,Q}) = E_Q^*$. Moreover, if we introduce the partial potentials

$$U_i^\mu = \sum_{j=1}^d c_{i,j} U^{\mu_j}, \quad i = 1, \dots, d,$$

it is proved in [3, Theorem 1.8] that a measure μ minimizes the weighted energy E_Q if and only if there exist constants F_1, \dots, F_d such that

$$U_i^\mu(z) + Q_i \geq F_i, \quad \text{q.e. } z \in K_i, \quad i = 1, \dots, d, \quad (2.1)$$

$$U_i^\mu(z) + Q_i \leq F_i, \quad \mu_i\text{-a.e. } z \in K_i, \quad i = 1, \dots, d. \quad (2.2)$$

3 Discretization of the vector energy

Throughout this section, we continue with the same assumptions as above:

1. $C \geq 0$ and C has no zero columns (or rows); $r_1, \dots, r_d > 0$;
2. K_1, \dots, K_d nonpolar with $K_i \cap K_j = \emptyset$, $i \neq j$;
3. Q_1, \dots, Q_d admissible.

To discretize the vector energies E and E_Q , for each $k = 1, 2, \dots$ we take a sequence of ordered tuples $m_k = (m_{1,k}, \dots, m_{d,k})$ of positive integers with

$$m_{i,k} \uparrow \infty, \quad i = 1, \dots, d, \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{m_{i,k}}{m_{j,k}} = \frac{r_i}{r_j}, \quad i, j = 1, \dots, d. \quad (3.1)$$

Note with this hypothesis

$$\frac{r_i^2}{m_{i,k}^2} \asymp \frac{r_j^2}{m_{j,k}^2} \asymp \frac{r_i r_j}{m_{i,k} m_{j,k}}, \quad (3.2)$$

where the notation $a_k \asymp b_k$ stands for asymptotically equal, i.e. $a_k/b_k \rightarrow 1$ as $k \rightarrow \infty$. For a set of distinct points of the form

$$\mathbf{Z}_k = \cup_{i=1}^d \{z_{i,1}, \dots, z_{i,m_{i,k}} \in K_i\}, \quad (3.3)$$

let

$$|VDM_k(\mathbf{Z}_k)| := \prod_{i=1}^d \prod_{l < p}^{m_{i,k}} |z_{i,l} - z_{i,p}|^{c_{i,i}} \cdot \prod_{i < j}^d \prod_{l=1}^{m_{i,k}} \prod_{p=1}^{m_{j,k}} |z_{i,l} - z_{j,p}|^{c_{i,j}}. \quad (3.4)$$

We define a k -th order vector diameter with respect to $(m_{1,k}, \dots, m_{d,k})$ – all that follows will be with respect to a sequence satisfying (3.1) – via

$$\delta^{(k)}(K) := \max_{\mathbf{Z}_k} \left[|VDM_k(\mathbf{Z}_k)| \right]^{2|r|^2/|m_k|(|m_k|-1)}, \quad (3.5)$$

where we set

$$|r| = r_1 + \dots + r_d, \quad |m_k| = m_{1,k} + \dots + m_{d,k}.$$

Given a weight $Q = (Q_1, \dots, Q_d)$, we define

$$|VDM_k^Q(\mathbf{Z}_k)| := |VDM_k(\mathbf{Z}_k)| \cdot \prod_{i=1}^d \prod_{l=1}^{m_{i,k}} e^{-\frac{m_{i,k}}{r_i} Q_i(z_{i,l})}$$

and we have the k -th order weighted vector diameter:

$$\delta_Q^{(k)}(K) := \max_{\mathbf{Z}_k} \left[|VDM_k^Q(\mathbf{Z}_k)| \right]^{2|r|^2/|m_k|(|m_k|-1)}. \quad (3.6)$$

Note that, similarly to the classical scalar case, the factor $|m_m|(|m_k|-1)/2$ in the exponent of (3.5) and (3.6) corresponds to the number of factors in the product (3.4). Actually the “ -1 ” in $|m_m|(|m_k|-1)/2$ could be dropped but with it the formulas reduce to those in the scalar case.

We start with a general result which will also be used in section 6. The proof is similar to the classical (scalar) case; cf., [19].

Proposition 3.1. *Take a sequence $\{(m_{1,k}, \dots, m_{d,k})\}$ satisfying (3.1) and $\mu = (\mu_1, \dots, \mu_d) \in \mathcal{M}_r(K)$. Let*

$$\mu^k = (\mu_1^k, \dots, \mu_d^k) := \left(\frac{r_1}{m_{1,k}} \sum_{j=1}^{m_{1,k}} \delta_{z_{1,j}^{(k)}}, \dots, \frac{r_d}{m_{d,k}} \sum_{j=1}^{m_{d,k}} \delta_{z_{d,j}^{(k)}} \right)$$

be a sequence of discrete measures in $\mathcal{M}_r(K)$ associated to the array

$$\mathbf{Z}_k = \cup_{i=1}^d \{z_{i,1}^{(k)}, \dots, z_{i,m_{i,k}}^{(k)} \in K_i\},$$

with $\mu^k \rightarrow \mu$ weak-. Then*

$$\limsup_{k \rightarrow \infty} |VDM_k(\mathbf{Z}_k)|^{2|r|^2/|m_k|(|m_k|-1)} \leq e^{-E(\mu)}. \quad (3.7)$$

In the weighted case,

$$\limsup_{k \rightarrow \infty} |VDM_k^Q(\mathbf{Z}_k)|^{2|r|^2/|m_k|(|m_k|-1)} \leq e^{-E_Q(\mu)}. \quad (3.8)$$

Proof. We have $\mu_i^k \times \mu_j^k \rightarrow \mu_i \times \mu_j$ weak-* for $i, j = 1, \dots, d$. Furthermore, the function $(a, b) \rightarrow \log \frac{1}{|a-b|}$ is lowersemicontinuous. For a real number M let

$$h_M(z, t) := \min[M, \log \frac{1}{|z-t|}] \leq \log \frac{1}{|z-t|}.$$

Then, for $i = 1, \dots, d$, we have

$$\begin{aligned} I(\mu_i) &= \lim_{M \rightarrow \infty} \int_{K_i} \int_{K_i} h_M(z, t) d\mu_i(z) d\mu_i(t) \\ &= \lim_{M \rightarrow \infty} \lim_{k \rightarrow \infty} \int_{K_i} \int_{K_i} h_M(z, t) d\mu_i^k(z) d\mu_i^k(t). \end{aligned}$$

Now

$$h_M(z_{i,l}^{(k)}, z_{i,p}^{(k)}) \leq \log \frac{1}{|z_{i,l}^{(k)} - z_{i,p}^{(k)}|},$$

if $l \neq p$ and hence

$$\int_{K_i} \int_{K_i} h_M(z, t) d\mu_i^k(z) d\mu_i^k(t) \leq \frac{r_i^2}{m_{i,k}^2} \left(m_{i,k} M + \sum_{l \neq p} \log \frac{1}{|z_{i,l}^{(k)} - z_{i,p}^{(k)}|} \right).$$

Consequently,

$$\begin{aligned} I(\mu_i) &\leq \lim_{M \rightarrow \infty} \liminf_{k \rightarrow \infty} \frac{r_i^2}{m_{i,k}^2} \left(m_{i,k} M + \sum_{l \neq p} \log \frac{1}{|z_{i,l}^{(k)} - z_{i,p}^{(k)}|} \right) \\ &= \liminf_{k \rightarrow \infty} \frac{r_i^2}{m_{i,k}^2} \sum_{l \neq p} \log \frac{1}{|z_{i,l}^{(k)} - z_{i,p}^{(k)}|}. \end{aligned} \quad (3.9)$$

Finally, since $K_i \cap K_j = \emptyset$, $i \neq j$, from $\mu_i^k \times \mu_j^k \rightarrow \mu_i \times \mu_j$ weak-* we have

$$I(\mu_i, \mu_j) = \lim_{k \rightarrow \infty} I(\mu_i^k, \mu_j^k) = \lim_{k \rightarrow \infty} \frac{r_i r_j}{m_{i,k} m_{j,k}} \sum_{l=1}^{m_{i,k}} \sum_{p=1}^{m_{j,k}} \log \frac{1}{|z_{i,l}^{(k)} - z_{j,p}^{(k)}|}. \quad (3.10)$$

Putting estimates (3.9) and (3.10) for $i, j = 1, \dots, d$, together gives

$$\limsup_{k \rightarrow \infty} \sum_{i=1}^d \frac{r_i^2}{m_{i,k}^2} \sum_{l \neq p} \log |z_{i,l}^{(k)} - z_{i,p}^{(k)}|^{c_{i,i}} + \lim_{k \rightarrow \infty} \sum_{i \neq j} \frac{r_i r_j}{m_{i,k} m_{j,k}} \sum_{l=1}^{m_{i,k}} \sum_{p=1}^{m_{j,k}} \log |z_{i,l}^{(k)} - z_{j,p}^{(k)}|^{c_{i,j}} \leq -E(\mu).$$

Then, using (3.1) and (3.2) leads to

$$\limsup_{k \rightarrow \infty} \frac{2|r|^2}{|m_k|(|m_k| - 1)} \log |VDM_k(\mathbf{Z}_k)| \leq -E(\mu),$$

which proves (3.7).

The weighted case (3.8) follows from the unweighted case, (3.1) and (3.2), and lower semicontinuity of Q_1, \dots, Q_d . \square

Proposition 3.2. *In the unweighted case,*

$$\delta(K) := \lim_{k \rightarrow \infty} \delta^{(k)}(K) = e^{-E^*} = e^{-E(\mu^K)}$$

and in the weighted case,

$$\delta_Q(K) := \lim_{k \rightarrow \infty} \delta_Q^{(k)}(K) = e^{-E_Q^*} = e^{-E_Q(\mu^{K,Q})}.$$

Proof. We prove the unweighted case; the weighted case is similar. First observe that if we take any points

$$\mathbf{Z}_k = \cup_{i=1}^d \{z_{i,1}^{(k)}, \dots, z_{i,m_{i,k}}^{(k)} \in K_i\},$$

then

$$\begin{aligned} & -\frac{|m_k|(|m_k| - 1)}{2|r|^2} \log \delta^{(k)}(K) \leq -\log |VDM_k(\mathbf{Z}_k)| \\ &= \sum_{i=1}^d c_{i,i} \sum_{l < p}^{m_{i,k}} \log \frac{1}{|z_{i,l} - z_{i,p}|} + \sum_{i < j}^d c_{i,j} \sum_{l=1}^{m_{i,k}} \sum_{p=1}^{m_{j,k}} \log \frac{1}{|z_{i,l} - z_{j,p}|}. \end{aligned}$$

Given any $\sigma = (\sigma_1, \dots, \sigma_d) = (r_1 \bar{\sigma}_1, \dots, r_d \bar{\sigma}_d) \in \mathcal{M}_r(K)$ where $\bar{\sigma}_i \in \mathcal{M}_1(K_i)$, $i = 1, \dots, d$, if we integrate with respect to the probability measure

$$\prod_{i=1}^d \prod_{l < p}^{m_{i,k}} d\bar{\sigma}_i(z_{i,l}) d\bar{\sigma}_i(z_{i,p}) \cdot \prod_{i < j}^d \prod_{l=1}^{m_{i,k}} \prod_{p=1}^{m_{j,k}} d\bar{\sigma}_i(z_{i,l}) d\bar{\sigma}_j(z_{j,p})$$

we get

$$\begin{aligned} -\frac{|m_k|(|m_k| - 1)}{2|r|^2} \log \delta^{(k)}(K) &\leq \sum_{i=1}^d c_{i,i} \frac{m_{i,k}(m_{i,k} - 1)}{2} I(\bar{\sigma}_i) + \sum_{i < j} c_{i,j} m_{i,k} m_{j,k} I(\bar{\sigma}_i, \bar{\sigma}_j) \\ &= \sum_{i=1}^d c_{i,i} \frac{m_{i,k}(m_{i,k} - 1)}{2r_i^2} I(\sigma_i) + \sum_{i < j} c_{i,j} \frac{m_{i,k} m_{j,k}}{r_i r_j} I(\sigma_i, \sigma_j). \end{aligned}$$

Then we use (3.2) to obtain

$$e^{-E(\sigma)} \leq \liminf_{k \rightarrow \infty} \left(\delta^{(k)}(K) \right). \quad (3.11)$$

Next, let

$$\mathbf{Z}_k = \cup_{i=1}^d \{z_{i,1}^{(k)}, \dots, z_{i,m_{i,k}}^{(k)} \in K_i\},$$

be a Fekete array of order k ; i.e., achieving the maximum for $\delta^{(k)}(K)$ in (3.5). Letting $\mu = (\mu_1, \dots, \mu_d) \in \mathcal{M}_r(K)$ be any weak-* limit of the sequence of Fekete measures

$$\mu^k := \left(\frac{r_1}{m_{1,k}} \sum_{j=1}^{m_{1,k}} \delta_{z_{1,j}^{(k)}}, \dots, \frac{r_d}{m_{d,k}} \sum_{j=1}^{m_{d,k}} \delta_{z_{d,j}^{(k)}} \right),$$

Proposition 3.1 gives

$$\limsup_{k \rightarrow \infty} \left[\delta^{(k)}(K) \right] \leq e^{-E(\mu)}.$$

Thus, with (3.11),

$$E(\mu) \leq \lim_{k \rightarrow \infty} \left[-\log \delta^{(k)}(K) \right] \leq E(\sigma),$$

for any σ . Hence the limit exists and equals the energy of any weak-* limit μ of Fekete measures. Since there exists a unique minimizing measure in $\mathcal{M}_r(K)$ for E , we have $\mu = \mu^K$ and $\lim_{k \rightarrow \infty} \delta^{(k)}(K) = e^{-E(\mu^K)}$. \square

Note that our definition of the k -th order (weighted) diameter is relative to m_k , but the proof shows that the (weighted) transfinite diameter $\delta(K)$ ($\delta_Q(K)$) is independent of the sequence m_k satisfying (3.1).

The proof of Proposition 3.2 included the result that (weighted) Fekete measures μ^k converge weak-* to the (weighted) energy minimizing measure μ^K ($\mu^{K,Q}$). Indeed, the proof shows the result for *asymptotic* (weighted) Fekete measures:

Proposition 3.3. *In the unweighted case, for an array*

$$\mathbf{Z}_{\mathbf{k}} = \cup_{i=1}^d \{z_{i,1}^{(k)}, \dots, z_{i,m_{i,k}}^{(k)} \in K_i\},$$

if

$$\lim_{k \rightarrow \infty} |VDM_k(\mathbf{Z}_{\mathbf{k}})|^{2|r|^2/|m_k|(|m_k|-1)} = e^{-E^*}$$

then

$$\mu^k := \left(\frac{r_1}{m_{1,k}} \sum_{j=1}^{m_{1,k}} \delta_{z_{1,j}^{(k)}}, \dots, \frac{r_d}{m_{d,k}} \sum_{j=1}^{m_{d,k}} \delta_{z_{d,j}^{(k)}} \right) \rightarrow \mu^K \text{ weak-}^*$$

and in the weighted case, if

$$\lim_{k \rightarrow \infty} |VDM_k^Q(\mathbf{Z}_{\mathbf{k}})|^{2|r|^2/|m_k|(|m_k|-1)} = e^{-E_Q^*}$$

then

$$\mu^k := \left(\frac{r_1}{m_{1,k}} \sum_{j=1}^{m_{1,k}} \delta_{z_{1,j}^{(k)}}, \dots, \frac{r_d}{m_{d,k}} \sum_{j=1}^{m_{d,k}} \delta_{z_{d,j}^{(k)}} \right) \rightarrow \mu^{K,Q} \text{ weak-}^*.$$

Proof. We prove the unweighted case; the weighted case is similar. Let $\sigma = (\sigma_1, \dots, \sigma_d) \in \mathcal{M}_r(K)$ be any weak- * limit of the sequence of measures μ^k . The proof of Proposition 3.2 shows that

$$E^* = \limsup_{k \rightarrow \infty} \left(-\log \delta^{(k)}(K) \right) \leq E(\sigma);$$

then Proposition 3.1 gives

$$E(\sigma) \leq \liminf_{k \rightarrow \infty} \left[\frac{-2|r|^2}{|m_k|(|m_k|-1)} \log |VDM_k(\mathbf{Z}_{\mathbf{k}})| \right] = E^*.$$

Thus

$$E^* = E(\sigma)$$

so that σ minimizes E over all $\mu \in \mathcal{M}_r(K)$. Since there exists a unique minimizer for E , we are done. \square

Again, if we need to keep track of the underlying interaction matrix C , we write

$$-\log \delta^C(K) = (E^C)^* \text{ and } -\log \delta_Q^C(K) = (E_Q^C)^*.$$

Occasionally we may write $VDM_k^{(C)}$ as well. If $\alpha \in \mathbb{C} \setminus \{0\}$, then $K_i \cap K_j = \emptyset$ implies $\alpha K_i \cap \alpha K_j = \emptyset$ and we have, using the definitions of $\delta^{(k)}(K)$ and $\delta_Q^{(k)}(K)$ together with (3.1) and (3.2), the scaling relations

$$\delta^C(\alpha K) = |\alpha|^B \delta^C(K) \text{ and } \delta_Q^C(\alpha K) = |\alpha|^B \delta_Q^C(K) \quad (3.12)$$

where

$$B = B(C, r) = \sum_{i,j=1}^d c_{i,j} r_i r_j \geq 0.$$

Note that B is independent of the sequence m_k used to define the k -th order diameters.

We use Proposition 3.2 and (3.12) to prove an important continuity property of the (weighted) vector transfinite diameter.

Proposition 3.4. *Given $C := (c_{i,j})_{i,j=1}^d$ we can find $C^{(k)} := (c_{i,j}^{(k)})_{i,j=1}^d$ symmetric positive semidefinite with all entries $c_{i,j}^{(k)}$ rational, $C^{(k)} \rightarrow C$ componentwise, and*

$$\lim_{k \rightarrow \infty} \delta^{C^{(k)}}(K) = \delta^C(K) \text{ and } \lim_{k \rightarrow \infty} \delta_Q^{C^{(k)}}(K) = \delta_Q^C(K).$$

Proof. From Proposition 3.2, we can instead work with the (weighted) minimal energies. We first prove the unweighted case. We take $c_{i,j}^{(k)}$ rational with $c_{i,j}^{(k)} \downarrow c_{i,j}$ for $c_{i,j} \geq 0$ and $c_{i,j}^{(k)} \uparrow c_{i,j}$ for $c_{i,j} < 0$. Note that, by choosing $|c_{i,j} - c_{i,j}^{(k)}|$, $i \neq j$, sufficiently small with respect to $c_{i,i}^{(k)} - c_{i,i}$, $i = 1, \dots, d$, the matrix $C^{(k)}$ is symmetric positive semidefinite.

Let $\mu^K = (\mu_1, \dots, \mu_d)$ satisfy $E^C(\mu^K) = (E^C)^*$. By rescaling (see (3.12)), we may assume K_1, \dots, K_d are contained in a disk of radius $1/2$ so that all energies $I(\mu_i)$ and $I(\mu_i, \mu_j)$ are nonnegative. Then

$$(E^C)^* \leq (E^{C^{(k)}})^* \leq E^{C^{(k)}}(\mu^K).$$

Now, simply by continuity, since $C^{(k)} \rightarrow C$, given $\epsilon > 0$,

$$|E^{C^{(k)}}(\mu^K) - E^C(\mu^K)| < \epsilon$$

for k sufficiently large and the result follows.

For the weighted case, let $\mu^{K,Q}$ satisfy $E_Q^C(\mu^{K,Q}) = (E_Q^C)^*$. Again from (3.12) we can assume all K_i are contained in a disk of radius $1/2$ and we have the similar inequality

$$(E_Q^C)^* \leq (E_Q^{C^{(k)}})^* \leq E_Q^{C^{(k)}}(\mu^{K,Q}).$$

The proof proceeds as in the unweighted case. □

4 Bernstein-Markov properties

In the first subsection, we define the notion of *strong rational Bernstein-Markov property* and we show that on any nonpolar compact set of \mathbb{C} there exists a positive measure that satisfies such a property. In the second subsection, we define a vector analog of this notion and we use it to show that the L^2 versions of the k -th order vector diameters defined in (3.5) and (3.6) have the same asymptotic behavior as k tends to infinity.

4.1 Bernstein-Markov properties in \mathbb{C}^n

For any $n = 1, 2, \dots$, let $\mathcal{P}_k = \mathcal{P}_k^{(n)}$ denote the holomorphic polynomials in n variables of degree at most k . Given a compact set $K \subset \mathbb{C}^n$ and a measure ν on K , we say that (K, ν) satisfies a Bernstein-Markov property if for all $p_k \in \mathcal{P}_k$,

$$\|p_k\|_K := \sup_{z \in K} |p_k(z)| \leq M_k \|p_k\|_{L^2(\nu)} \text{ with } \limsup_{k \rightarrow \infty} M_k^{1/k} = 1.$$

We will need to use the Bernstein-Markov property in \mathbb{C}^2 to derive properties in the univariate case. It was shown in [9] that any compact set in \mathbb{C}^n admits a Bernstein-Markov measure; indeed, the following stronger statement is true.

Proposition 4.1 ([9]). *Let $K \subset \mathbb{R}^n$. There exists a measure $\nu \in \mathcal{M}(K)$ such that for all complex-valued polynomials p of degree at most k in the (real) coordinates $x = (x_1, \dots, x_n)$ we have*

$$\|p\|_K \leq M_k \|p\|_{L^2(\nu)}$$

where $\limsup_{k \rightarrow \infty} M_k^{1/k} = 1$.

More generally, for $K \subset \mathbb{C}^n$ compact, Q admissible (Q is lowersemicontinuous and finite on a nonpluripolar set), and ν a measure on K , we say that the triple (K, ν, Q) satisfies a weighted Bernstein-Markov property if for all $p_k \in \mathcal{P}_k$,

$$\|e^{-kQ} p_k\|_K \leq M_k \|e^{-kQ} p_k\|_{L^2(\nu)} \text{ with } \limsup_{k \rightarrow \infty} M_k^{1/k} = 1.$$

Here K should be nonpluripolar for this notion to have any content. For the definition of pluripolar, the \mathbb{C}^n -analogue of polar, see Appendix B of [19].

Remark 4.2. An important observation is the following. If (K, ν, Q) satisfies a weighted Bernstein-Markov property for some admissible weight Q on K , then for any sequence $\{Q_k\}$ of admissible weights on K which converges uniformly to Q on K , we have a “varying weight” Bernstein-Markov property:

$$\lim_{k \rightarrow \infty} \left(\sup_{p_k \in \mathcal{P}_k} \frac{\|e^{-kQ_k} p_k\|_K}{\|e^{-kQ_k} p_k\|_{L^2(\mu)}} \right)^{1/k} = 1. \quad (4.1)$$

To verify (4.1), note simply that given $\epsilon > 0$ we have

$$e^{-kQ} e^{-k\epsilon} < e^{-kQ_k} < e^{-kQ} e^{k\epsilon}$$

on all of K for k sufficiently large.

These properties can be stated using $L^p(\nu)$ in place of $L^2(\nu)$, but it is known that *if (K, ν) satisfies an (weighted) L^p -Bernstein-Markov property for some $0 < p < \infty$ then (K, ν) satisfies an (weighted) L^p -Bernstein-Markov property for all $0 < p < \infty$* . This follows, for example, from Remark 3.2 in [7]; see also the proof of Theorem 3.4.3 in [20]. Thus, we simply say that (K, ν) satisfies a (weighted) Bernstein-Markov property.

Definition 4.3. We say (K, ν) satisfies a *strong Bernstein-Markov property* if (K, ν, Q) satisfies a weighted Bernstein-Markov property for each *continuous* Q .

Again, K should be nonpluripolar for this notion to have any content.

Now we return to $n = 1$; i.e., \mathbb{C} , and we next give a definition of a “rational” (weighted) Bernstein-Markov property, analogous to the definition for polynomials and for which the proof that this property being valid for some $p > 0$ implies it is valid for all $p > 0$ remains true. The paper [10] also concerns a rational Bernstein-Markov property. Given $K \subset \mathbb{C}$ compact, we fix a compact set K' disjoint from K and define, for $a, b > 0$,

$$\mathcal{R}_k = \{r_k = p_k/q_k : p_k, q_k \text{ polynomials; } \deg p_k \leq ak, \deg q_k \leq bk; \text{ all zeros of } q_k \text{ in } K'\}. \quad (4.2)$$

We say that (K, ν) satisfies a *rational Bernstein-Markov property* if for all $r_k \in \mathcal{R}_k$,

$$\|r_k\|_K := \sup_{z \in K} |r_k(z)| \leq M_k \|r_k\|_{L^2(\nu)} \text{ with } \limsup_{k \rightarrow \infty} M_k^{1/k} = 1.$$

Here $\mathcal{R}_k = \mathcal{R}_k(K', a, b)$. Note that taking $q_k \equiv 1$ we see that (K, ν) satisfies a (polynomial) Bernstein-Markov property.

More generally, for $K \subset \mathbb{C}$ compact, Q admissible, and ν a measure on K , we say that the triple (K, ν, Q) satisfies a weighted rational Bernstein-Markov property if for all $r_k \in \mathcal{R}_k$,

$$\|e^{-kQ} r_k\|_K \leq M_k \|e^{-kQ} r_k\|_{L^2(\nu)} \text{ with } \limsup_{k \rightarrow \infty} M_k^{1/k} = 1.$$

Definition 4.4. We say (K, ν) satisfies a *strong rational Bernstein-Markov property* if (K, ν, Q) satisfies a weighted rational Bernstein-Markov property for each *continuous* Q .

In the definitions of these various rational Bernstein-Markov properties, there is an implicit underlying pole set K' as well as positive numbers a, b . We will specify K', a, b in our vector setting in subsection 4.2.

To define certain vector energy functionals in section 6, and for our large deviation principle in section 7, we will need to use measures satisfying vector versions of the strong (rational) Bernstein-Markov property. We next prove that such measures always exist on nonpolar compacta in the scalar case; it will be clear from the proof that the constructed measures work for any fixed pole set K' and positive numbers a, b . For simplicity we take $b = 1$.

Theorem 4.5. *Let $K \subset \mathbb{C}$ be nonpolar. Then there exists ν on K with (K, ν) satisfying a strong rational Bernstein-Markov property.*

Proof. We consider $K \subset \mathbb{C} = \mathbb{R}^2 \subset \mathbb{C}^2$ with variables (z_1, z_2) where $\operatorname{Re} z_1 = x$ and $\operatorname{Re} z_2 = y$ so that $z = x + iy$ is the usual complex variable when we consider $\mathbb{C} = \mathbb{R}^2$. Using Proposition 4.1, we construct a measure ν on K such that (K, ν) satisfies a Bernstein-Markov property with respect to holomorphic polynomials on \mathbb{C}^2 . Theorem 3.2 of [8] then shows that (K, ν, Q) satisfies a weighted Bernstein-Markov property with respect to

holomorphic polynomials on \mathbb{C}^2 for all $Q \in C(K)$; i.e., (K, ν) satisfies a strong Bernstein-Markov property with respect to holomorphic polynomials on \mathbb{C}^2 . Since a holomorphic polynomial in z of degree at most n is of the form

$$p_n(z) = \sum a_j z^j = \sum a_j (x + iy)^j = \sum c_{kl} x^k y^l = \sum c_{kl} (\operatorname{Re} z_1)^k (\operatorname{Re} z_2)^l$$

where c_{kl} are complex numbers and $k + l \leq n$, each such p_n is the restriction to \mathbb{R}^2 of a holomorphic polynomial $\tilde{p}_n(z_1, z_2) := \sum c_{kl} z_1^k z_2^l$ in \mathbb{C}^2 . Thus (K, ν) satisfies a strong Bernstein-Markov property with respect to holomorphic polynomials on \mathbb{C} .

Applying Remark 4.2, (K, ν) satisfies a “varying weight” Bernstein-Markov property for any continuous target weight: for any $Q \in C(K)$, and any sequence $\{Q_k\}$ of admissible weights on K which converges uniformly to Q on K , (4.1) holds:

$$\lim_{k \rightarrow \infty} \left(\sup_{p_k \in \mathcal{P}_k} \frac{\|e^{-kQ_k} p_k\|_K}{\|e^{-kQ_k} p_k\|_{L^2(\mu)}} \right)^{1/k} = 1.$$

We now fix $Q \in C(K)$ and consider the sequence of numbers

$$\left\{ \left(\sup_{r_n \in \mathcal{R}_n} \frac{\|e^{-nQ} r_n\|_K}{\|e^{-nQ} r_n\|_{L^2(\mu)}} \right)^{1/n} \right\}.$$

Let

$$\alpha := \limsup_{n \rightarrow \infty} \left(\sup_{r_n \in \mathcal{R}_n} \frac{\|e^{-nQ} r_n\|_K}{\|e^{-nQ} r_n\|_{L^2(\mu)}} \right)^{1/n}.$$

Clearly $\alpha \geq 1$; we want to show $\alpha = 1$. Take a subsequence $\{n_k\}$ of integers so that

$$\lim_{k \rightarrow \infty} \left(\sup_{r_{n_k} \in \mathcal{R}_{n_k}} \frac{\|e^{-n_k Q} r_{n_k}\|_K}{\|e^{-n_k Q} r_{n_k}\|_{L^2(\mu)}} \right)^{1/n_k} = \alpha$$

and, given $\epsilon > 0$, choose $r_{n_k} \in \mathcal{R}_{n_k}$ with

$$\left(\frac{\|e^{-n_k Q} r_{n_k}\|_K}{\|e^{-n_k Q} r_{n_k}\|_{L^2(\mu)}} \right)^{1/n_k} \geq \left(\sup_{r_{n_k} \in \mathcal{R}_{n_k}} \frac{\|e^{-n_k Q} r_{n_k}\|_K}{\|e^{-n_k Q} r_{n_k}\|_{L^2(\mu)}} \right)^{1/n_k} - \epsilon.$$

Writing $r_{n_k} := p_{n_k}/q_{n_k}$ where we take $q_{n_k} = \prod_{j=1}^{n_k} (z - z_j^{(k)})$ monic with zeros in K' , we have

$$e^{-n_k Q} |r_{n_k}| = \frac{e^{-n_k Q}}{|q_{n_k}|} \cdot |p_{n_k}| =: e^{-n_k Q_{n_k}} \cdot |p_{n_k}|$$

where

$$e^{-n_k Q_{n_k}} = \frac{e^{-n_k Q}}{|q_{n_k}|} \text{ so that } Q_{n_k} = Q + \frac{1}{n_k} \log |q_{n_k}|.$$

Now $(-1/n_k) \log |q_{n_k}|$ is the logarithmic potential U^{μ_k} of the probability measure

$$\mu_k := \frac{1}{n_k} \sum_{j=1}^{n_k} \delta_{z_j^{(k)}}$$

which is supported in K' . Taking a weak-* limit of this sequence $\{\mu_k\}$ we get a probability measure ν on K' with $U^{\mu_k} \rightarrow U^\nu$ uniformly on K ; hence, taking the corresponding subsequence of $\{n_k\}$ (which we do not relabel) we have

$$Q_{n_k} \rightarrow Q - U^\nu \text{ uniformly on } K.$$

Note that U^ν is harmonic and hence continuous on K . We extend the definition of Q_n for $n \notin \{n_k\}$ by simply defining $Q_n := Q - U^\nu$ for such n . Then the full sequence $\{Q_n\}$ satisfies $Q_n \rightarrow Q - U^\nu$ uniformly on K and thus we have from (4.1) that

$$\lim_{n \rightarrow \infty} \left(\sup_{p_n \in \mathcal{P}_n} \frac{\|e^{-nQ_n} p_n\|_K}{\|e^{-nQ_n} p_n\|_{L^2(\mu)}} \right)^{1/n} = 1.$$

But for $n = n_k$ we have

$$\begin{aligned} \left(\sup_{p_{n_k} \in \mathcal{P}_{n_k}} \frac{\|e^{-n_k Q_{n_k}} p_{n_k}\|_K}{\|e^{-n_k Q_{n_k}} p_{n_k}\|_{L^2(\mu)}} \right)^{1/n_k} &\geq \left(\frac{\|e^{-n_k Q} r_{n_k}\|_K}{\|e^{-n_k Q} r_{n_k}\|_{L^2(\mu)}} \right)^{1/n_k} \\ &\geq \left(\sup_{r_{n_k} \in \mathcal{R}_{n_k}} \frac{\|e^{-n_k Q} r_{n_k}\|_K}{\|e^{-n_k Q} r_{n_k}\|_{L^2(\mu)}} \right)^{1/n_k} - \epsilon. \end{aligned}$$

Thus

$$\alpha = \lim_{k \rightarrow \infty} \left(\sup_{r_{n_k} \in \mathcal{R}_{n_k}} \frac{\|e^{-n_k Q} r_{n_k}\|_K}{\|e^{-n_k Q} r_{n_k}\|_{L^2(\mu)}} \right)^{1/n_k} = 1.$$

□

Remark 4.6. There are easy-to-check sufficient conditions for a measure to satisfy a strong (rational) Bernstein-Markov property. Let $K \subset \mathbb{C}^n$. We say (K, ν) satisfies a *mass-density property* if there exists $T > 0$ with $\nu(B(z_0, r)) \geq r^T$ for all $z_0 \in K$ and all $r < r(z_0)$ where $B(z_0, r)$ is the ball of radius r centered at z_0 . For K regular in the pluripotential-theoretic sense (see Appendix B of [19]), this property implies that (K, ν) satisfies a Bernstein-Markov property; hence if $K \subset \mathbb{R}^2 \subset \mathbb{C}^2$ has this regularity and (K, ν) satisfies a mass-density property, then the proof of Theorem 4.5 shows that (K, ν) satisfies a strong rational Bernstein-Markov property. In particular, if $K = \bar{D}$ when D is a bounded domain in \mathbb{R}^2 with C^1 -boundary, any ν which is a positive, continuous multiple of Lebesgue measure on D is a strong rational Bernstein-Markov measure for K .

4.2 Vector Bernstein-Markov property

Definition 4.7. Let $0 < p < \infty$ and let $\nu = (\nu_1, \dots, \nu_d)$ be a tuple of measures with ν_i supported in K_i for $i = 1, \dots, d$. Recall $K = (K_1, \dots, K_d)$. We say (K, ν) satisfies an *L^p -Bernstein-Markov property* if for $i = 1, \dots, d$,

$$\|p_k\|_{K_i} \leq M_{k,i}^{(p)} \|p_k\|_{L^p(\nu_i)}, \quad p_k \in \mathcal{P}_k$$

where $(M_{k,i}^{(p)})^{1/k} \rightarrow 1$ as $k \rightarrow \infty$; i.e., each (K_i, ν_i) satisfies an L^p -Bernstein-Markov property.

It follows from the scalar case that if (K, ν) satisfies an L^p -Bernstein-Markov property for some $0 < p < \infty$ then (K, ν) satisfies an L^p -Bernstein-Markov property for all $0 < p < \infty$. Thus, we simply say, in our vector setting, that (K, ν) satisfies a Bernstein-Markov property.

Now let $Q = (Q_1, \dots, Q_d)$ be a d -tuple of admissible weights for $K = (K_1, \dots, K_d)$.

Definition 4.8. We say (K, ν, Q) satisfies an L^p -weighted Bernstein-Markov property if for $i = 1, \dots, d$,

$$\|p_k e^{-kQ_i}\|_{K_i} \leq M_{k,i}^{(p)} \|p_k e^{-kQ_i}\|_{L^p(\nu_i)}, \quad p_k \in \mathcal{P}_k$$

where $(M_{k,i}^{(p)})^{1/k} \rightarrow 1$ as $k \rightarrow \infty$; i.e., each (K_i, ν_i, Q_i) satisfies an L^p -weighted Bernstein-Markov property.

Definition 4.9. We say (K, ν) satisfies a strong Bernstein-Markov property if (K, ν, Q) satisfies a weighted Bernstein-Markov property for each continuous Q .

We next define vector versions of rational Bernstein-Markov properties. Our setting is the following: the classes \mathcal{R}_k defined in (4.2) will be taken with $K = K_i$ and $K' = \cup_{j \neq i} K_j$, for $i = 1, \dots, d$:

$$\mathcal{R}_k^i = \{r_k = p_k/q_k : p_k, q_k \text{ polynomials; } \deg p_k \leq ak, \deg q_k \leq bk; \text{ all zeros of } q_k \text{ in } \cup_{j \neq i} K_j\}.$$

Given an interaction matrix $C \geq 0$ and $r_1, \dots, r_d > 0$, the a, b we choose will depend on the coefficients $c_{i,j}$ of C as well as r_1, \dots, r_d .

Definition 4.10. Let $0 < p < \infty$ and let $\nu = (\nu_1, \dots, \nu_d)$ be a tuple of measures with ν_i supported in K_i for $i = 1, \dots, d$. We say (K, ν) satisfies an L^p -rational Bernstein-Markov property if for $i = 1, \dots, d$,

$$\|r_k\|_{K_i} \leq M_{k,i}^{(p)} \|r_k\|_{L^p(\nu_i)}, \quad r_k \in \mathcal{R}_k^i$$

where $(M_{k,i}^{(p)})^{1/k} \rightarrow 1$ as $k \rightarrow \infty$; i.e., each (K_i, ν_i) satisfies an L^p -rational Bernstein-Markov property.

From the scalar setting again we simply say that (K, ν) satisfies a rational Bernstein-Markov property since the property holds for all $p > 0$ once it holds for any $p > 0$. Also, as in the scalar case, if (K, ν) satisfies a rational Bernstein-Markov property then (K, ν) satisfies a (polynomial) Bernstein-Markov property.

Definition 4.11. For $Q = (Q_1, \dots, Q_d)$, we say (K, ν, Q) satisfies an L^p -weighted rational Bernstein-Markov property if for $i = 1, \dots, d$,

$$\|r_k e^{-kQ_i}\|_{K_i} \leq M_{k,i}^{(p)} \|r_k e^{-kQ_i}\|_{L^p(\nu_i)}, \quad r_k \in \mathcal{R}_k^i$$

where $(M_{k,i}^{(p)})^{1/k} \rightarrow 1$ as $k \rightarrow \infty$; i.e., each (K_i, ν_i, Q_i) satisfies an L^p -weighted rational Bernstein-Markov property.

Definition 4.12. We say (K, ν) satisfies a *strong rational Bernstein-Markov property* if (K, ν, Q) satisfies a weighted rational Bernstein-Markov property for each *continuous* Q .

Appealing to the scalar case result that any nonpolar compact set $K \subset \mathbb{C}$ admits a measure μ such that (K, μ) satisfies a strong rational Bernstein-Markov property (Theorem 4.5), we thus have the analogous result in the vector case: any nonpolar tuple $K = (K_1, \dots, K_d)$ admits a strong rational Bernstein-Markov tuple $\nu = (\nu_1, \dots, \nu_d)$.

Remark 4.13. First a word on notation: given a sequence $\{m_k\}$ satisfying (3.1) and a sequence $\{\mathbf{Z}_k\}$ of points of the form (3.3), we write, with abuse of notation, $K^k := K_1^{m_{1,k}} \times \dots \times K_d^{m_{d,k}}$, and

$$d\nu(\mathbf{Z}_k) := d\nu_1(z_{1,1}) \dots d\nu_1(z_{1,m_{1,k}}) d\nu_2(z_{2,1}) \dots d\nu_d(z_{d,m_{d,k}}).$$

Next, given $C \geq 0$ and $r_1, \dots, r_d > 0$ in our vector energy setting, when we write “Bernstein-Markov property” below – and essentially for the rest of the paper – we will mean “polynomial Bernstein-Markov property” if all coefficients $c_{i,j}$ of C are nonnegative and “rational Bernstein-Markov property” otherwise.

Proposition 4.14. *Let $\{m_k\}$ be a sequence satisfying (3.1) and \mathbf{Z}_k a set of points of the form (3.3). Assume (K, ν) satisfies a Bernstein-Markov property. Let*

$$Z_k := \int_{K^k} |VDM_k(\mathbf{Z}_k)|^2 d\nu(\mathbf{Z}_k).$$

Then

$$\lim_{k \rightarrow \infty} Z_k^{|r|^2/|m_k|(|m_k|-1)} = e^{-E^*} = \delta^C(K).$$

In the weighted case, if (K, ν, Q) satisfies a weighted Bernstein-Markov property and

$$Z_k^Q := \int_{K^k} |VDM_k^Q(\mathbf{Z}_k)|^2 d\nu(\mathbf{Z}_k),$$

then

$$\lim_{k \rightarrow \infty} (Z_k^Q)^{|r|^2/|m_k|(|m_k|-1)} = e^{-E_Q^*} = \delta_Q^C(K).$$

Proof. We prove the unweighted version; the weighted version is similar. Clearly

$$Z_k^{|r|^2/|m_k|(|m_k|-1)} \leq \delta^{(k)}(K) [\nu(K^k)]^{|r|^2/|m_k|(|m_k|-1)},$$

and by letting $k \rightarrow \infty$,

$$\limsup_{k \rightarrow \infty} Z_k^{|r|^2/|m_k|(|m_k|-1)} \leq \delta^C(K).$$

Recall that

$$|VDM_k(\mathbf{Z}_k)| := \prod_{i=1}^d \prod_{l < p}^{m_{i,k}} |z_{i,l} - z_{i,p}|^{c_{i,i}} \cdot \prod_{i < j}^d \prod_{l=1}^{m_{i,k}} \prod_{p=1}^{m_{j,k}} |z_{i,l} - z_{j,p}|^{c_{i,j}}.$$

Case I: All coefficients $c_{i,j}$ are integers.

It is easily checked that $VDM_k(\mathbf{Z}_k)$ is a rational function whose numerator and denominator degrees are bounded by

$$\max_i \left(\sum_{j=1}^d m_{j,k} |c_{i,j}| \right) \leq A |m_k|$$

in each variable where $A = A(C) = \max(|c_{i,j}|)$.

Let $\mathbf{A}_k = (a_{1,1}, \dots, a_{d,m_{d,k}})$ be a set of Fekete points of order k for K . Then

$$p(z_{1,1}) := VDM_k(z_{1,1}, a_{1,2}, \dots, a_{d,m_{d,k}})$$

is a rational function in z_1 of numerator and denominator degrees at most $A|m_k|$ achieving its supremum norm on K_1 at $z_{1,1} = a_{1,1}$. By the Bernstein-Markov property, we have

$$|VDM_k(\mathbf{A}_k)|^2 \leq M_{|m_k|}^2 \int_{K_1} |VDM_k(z_{1,1}, a_{1,2}, \dots, a_{d,m_{d,k}})|^2 d\nu_1(z_{1,1}).$$

Now for each fixed $z_{1,1} \in K_1$, we consider

$$q(z_{1,2}) := VDM_k(z_{1,1}, z_{1,2}, \dots, a_{d,m_{d,k}})$$

as a rational function in $z_{1,2}$ of numerator and denominator degrees at most $A|m_k|$. Again, by the Bernstein-Markov property, we have

$$|q(a_{1,2})|^2 \leq \|q\|_{K_1}^2 \leq M_{|m_k|}^2 \int_{K_1} |q(z_{1,2})|^2 d\nu_1(z_{1,2}).$$

Inserting this in the integrand of our previous estimate gives

$$|VDM_k(\mathbf{A}_k)|^2 \leq M_{|m_k|}^4 \int_{K_1} \int_{K_1} |VDM_k(z_{1,1}, z_{1,2}, \dots, a_{d,m_{d,k}})|^2 d\nu_1(z_{1,1}) d\nu_1(z_{1,2}).$$

Continuing in this way, we obtain

$$|VDM_k(\mathbf{A}_k)|^2 \leq M_{|m_k|}^{2|m_k|} Z_k.$$

This says that

$$\delta^{(k)}(K) \leq M_{|m_k|}^{2|r|^2/(|m_k|-1)} Z_k^{|r|^2/|m_k|(|m_k|-1)}$$

and we are done since $M_{|m_k|}^{1/(|m_k|-1)} \rightarrow 1$ as $k \rightarrow \infty$.

Case II: All coefficients $c_{i,j}$ are rational numbers.

Let M be a positive integer such that each $M c_{i,j}$ is an integer. Now

$$p(z_{1,1}) := VDM_k(z_{1,1}, a_{1,2}, \dots, a_{d,m_{d,k}})^M$$

is a rational function in $z_{1,1}$ of numerator and denominator degrees at most $AM|m_k|$ achieving its supremum norm on K_1 at $z_{1,1} = a_{1,1}$. Applying the L^p -Bernstein-Markov property to this rational function with exponent $p = 2/M$ we have

$$\begin{aligned} |VDM_k(\mathbf{A}_k)|^2 &= \left(|VDM_k(\mathbf{A}_k)|^M \right)^{2/M} \\ &\leq \left(M_{|m_k|}^{(2/M)} \right)^{2/M} \int_{K_1} |VDM_k(z_{1,1}, a_{1,2}, \dots, a_{d,m_{d,k}})|^2 d\nu_1(z_1). \end{aligned}$$

For each fixed $z_{1,1} \in K_1$, we now consider

$$q(z_{1,2}) := VDM_k(z_{1,1}, z_{1,2}, \dots, a_{d,m_{d,k}})^M$$

as a rational function in $z_{1,2}$ of degree at most $AM|m_k|$. We have

$$\begin{aligned} |VDM_k(z_{1,1}, a_{1,2}, \dots, a_{d,m_{d,k}})|^2 &= |q(a_{1,2})|^{2/M} \leq \|q\|_{K_1}^{2/M} \\ &\leq \left(M_{|m_k|}^{(2/M)} \right)^{2/M} \int_{K_1} |q(z_{1,2})|^{2/M} d\nu_1(z_{1,2}) \\ &= \left(M_{|m_k|}^{(2/M)} \right)^{2/M} \int_{K_1} |VDM_k(z_{1,1}, z_{1,2}, \dots, a_{d,m_{d,k}})|^2 d\nu_1(z_{1,2}). \end{aligned}$$

Inserting this in the integrand of our previous estimate gives

$$|VDM_k(\mathbf{A}_k)|^2 \leq \left(M_{|m_k|}^{(2/M)} \right)^{4/M} \int_{K_1} \int_{K_1} |VDM_k(z_{1,1}, z_{1,2}, \dots, a_{d,m_{d,k}})|^2 d\nu_1(z_{1,1}) d\nu_1(z_{1,2}).$$

Continuing in this way, we obtain our result.

Case III: All coefficients $c_{i,j}$ are real numbers.

This case will follow from the previous case and Proposition 3.4. We can assume that K_1, \dots, K_d are contained in a disk of radius $1/2$ so that all factors $|z_{i,l} - z_{j,p}| \leq 1$, $i, j = 1, \dots, d$, $l = 1, \dots, m_{i,k}$, $p = 1, \dots, m_{j,k}$. Then for any \widehat{C} with rational entries $\widehat{c}_{i,j}$ as in the proof of Proposition 3.4, we have

$$|VDM_k^{(\widehat{C})}(\mathbf{Z}_k)| \leq |VDM_k^{(C)}(\mathbf{Z}_k)|, \quad (4.3)$$

for recall that $\widehat{c}_{i,j} \downarrow c_{i,j}$ if $c_{i,j} \geq 0$ and $\widehat{c}_{i,j} \uparrow c_{i,j}$ if $c_{i,j} < 0$. Hence

$$\delta^{\widehat{C}}(K) = \lim_{k \rightarrow \infty} \widehat{Z}_k^{\frac{|r|^2}{|m_k|(|m_k|-1)}} \leq \liminf_{k \rightarrow \infty} Z_k^{\frac{|r|^2}{|m_k|(|m_k|-1)}} \leq \limsup_{k \rightarrow \infty} Z_k^{\frac{|r|^2}{|m_k|(|m_k|-1)}} \leq \delta^C(K).$$

From Proposition 3.4 we have

$$\lim_{\widehat{C} \rightarrow C} \delta^{\widehat{C}}(K) = \delta^C(K)$$

which finishes the proof in the unweighted case. \square

Fix a tuple of weights Q . Given ν as in Proposition 4.14, i.e., so that (K, ν, Q) satisfies a weighted Bernstein-Markov property, and given a sequence $\{m_k\}$ satisfying (3.1), define a probability measure $Prob_k$ on K^k : for a Borel set $A \subset K^k$,

$$Prob_k(A) := \frac{1}{Z_k^Q} \cdot \int_A |VDM_k^Q(\mathbf{Z}_k)|^2 d\nu(\mathbf{Z}_k). \quad (4.4)$$

Directly from Proposition 4.14 and (4.4) we obtain the following estimate.

Corollary 4.15. *Let (K, ν, Q) satisfy a weighted Bernstein-Markov property. Given $\eta > 0$, define*

$$A_{k,\eta} := \{\mathbf{Z}_k \in K^k : |VDM_k^Q(\mathbf{Z}_k)|^2 \geq (\delta_Q(K) - \eta)^{|m_k|(|m_k|-1)/|r|^2}\}. \quad (4.5)$$

Then there exists $k^ = k^*(\eta)$ such that for all $k > k^*$,*

$$Prob_k(K^k \setminus A_{k,\eta}) \leq \left(1 - \frac{\eta}{2\delta_Q(K)}\right)^{|m_k|(|m_k|-1)/|r|^2} \nu(K^k).$$

We get the induced product probability measure \mathbf{P} on the space of arrays on K ,

$$\chi := \{X = \{\mathbf{Z}_k \in K^k\}_{k \geq 1}\},$$

namely,

$$(\chi, \mathbf{P}) := \prod_{k=1}^{\infty} (K^k, Prob_k).$$

As an immediate consequence of the Borel-Cantelli lemma, we obtain:

Corollary 4.16. *Let (K, ν, Q) satisfy a weighted Bernstein-Markov property. For \mathbf{P} -a.e. array $X \in \chi$,*

$$\mu^k = (\mu_1^k, \dots, \mu_d^k) := \left(\frac{r_1}{m_{1,k}} \sum_{j=1}^{m_{1,k}} \delta_{z_{1,j}^{(k)}}, \dots, \frac{r_d}{m_{d,k}} \sum_{j=1}^{m_{d,k}} \delta_{z_{d,j}^{(k)}} \right) \rightarrow \mu^{K,Q} \text{ weak-}^* \text{ as } k \rightarrow \infty.$$

Proof. From Proposition 3.3 it suffices to verify for \mathbf{P} -a.e. array $X = \{\mathbf{Z}_k\}_k \in \chi$,

$$\liminf_{k \rightarrow \infty} \left(|VDM_k^Q(\mathbf{Z}_k)| \right)^{2|r|^2/|m_k|(|m_k|-1)} = \delta_Q(K). \quad (4.6)$$

Given $\eta > 0$, the condition that for a given array $X = \{\mathbf{Z}_k\}_k$ we have

$$\liminf_{k \rightarrow \infty} \left(|VDM_k^Q(\mathbf{Z}_k)| \right)^{2|r|^2/|m_k|(|m_k|-1)} \leq \delta_Q(K) - \eta$$

means that $\mathbf{Z}_k \in K^k \setminus A_{k,\eta}$ for infinitely many k . Thus setting

$$E_k := \{X \in \chi : \mathbf{Z}_k \in K^k \setminus A_{k,\eta}\},$$

we have

$$\mathbf{P}(E_k) \leq \text{Prob}_k(K^k \setminus A_{k,\eta}) \leq (1 - \frac{\eta}{2\delta_Q(K)})^{|m_k|(|m_k|-1)/|r|^2} \nu(K^k),$$

whence $\sum_{k=1}^{\infty} \mathbf{P}(E_k) < +\infty$. By the Borel-Cantelli lemma,

$$\mathbf{P}(\limsup_{k \rightarrow \infty} E_k) = 0, \quad \text{where} \quad \limsup_{k \rightarrow \infty} E_k = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j.$$

Thus, with probability one, only finitely many E_k occur, and (4.6) follows. \square

5 Approximation of equilibrium problems with non-admissible weights

In Section 6, we will need to consider equilibrium problems with weights that are the negatives of potentials. These weights, if non-continuous, are non-admissible in the sense given in Section 2. The aim of this section is to show that one can approach such equilibrium problems by a sequence of equilibrium problems with continuous weights, see Lemma 5.2 for the scalar case and Lemma 5.4 for the vector case. In this section, K (or its component sets in the vector setting) will always be nonpolar.

Lemma 5.1. *Let $\mu \in \mathcal{M}_r(K)$, $K \subset \mathbb{C}$ compact, $I(\mu) < \infty$. Consider the possibly non-admissible weight $u := -U^\mu$ on K . The weighted minimal energy on K is obtained with the measure μ , that is*

$$\forall \nu \in \mathcal{M}_r(K), \quad I(\mu) + 2 \int u d\mu \leq I(\nu) + 2 \int u d\nu,$$

with equality if and only if $\nu = \mu$.

Proof. We may assume that $I(\nu) < \infty$. The inequality may be rewritten as

$$0 \leq I(\nu) - 2I(\mu, \nu) + I(\mu) = I(\nu - \mu),$$

which is true. Moreover, the energy $I(\nu - \mu)$ can vanish only when $\nu = \mu$ (cf., Lemma I.1.8 in [19]). \square

Lemma 5.2. *Let $K \subset \mathbb{C}$ be compact and nonpolar and let $\mu \in \mathcal{M}_r(K)$ with $I(\mu) < \infty$. There exist a sequence $\{K_n\}$ of compact subsets of K , a sequence of continuous functions Q_n on K , and a sequence $\{\mu_n\} \subset \mathcal{M}_r(K)$ such that*

1. each K_n is regular; $K_n \subset K_{n+1}$; and $\bigcup_n K_n = K \setminus P$ where P is polar;
2. $Q_n(z) \downarrow u(z) := -U^\mu(z)$, $z \in K$;

3. μ_n is the weighted energy minimizing measure over $\mathcal{M}_r(K_n)$ of $K_n, Q_n|_{K_n}$ and

$$\tilde{V}_{Q_n}(z) := -U^{\mu_n}(z) + F_n \downarrow u(z) := -U^\mu(z), \quad z \in \mathbb{C}$$

(defining the notation \tilde{V}_{Q_n} and the constant F_n) where \tilde{V}_{Q_n} and hence U^{μ_n} are continuous.

We have the following properties:

- (i) The Robin constants F_n tend to 0 as $n \rightarrow \infty$.
- (ii) The measures μ_n tend weak-* to μ , as $n \rightarrow \infty$.
- (iii) The energies $I(\mu_n)$ tend to $I(\mu)$ as $n \rightarrow \infty$.

Proof. Item 1. follows from Ancona's theorem [1]. Precisely, for each n we can find $\tilde{K}_n \subset K$ regular with $\text{Cap}(K \setminus \tilde{K}_n) < 1/n$; then $K_n := \cup_1^n \tilde{K}_j$ work. For 2., the function u is usc whence the existence of a monotone sequence of continuous functions Q_n decreasing to u on K .

To prove 3., note first that $Q_n|_{K_n}$ are continuous and K_n are regular so \tilde{V}_{Q_n} are continuous on \mathbb{C} (cf., Theorem I.5.1 in [19]). Since Q_n is decreasing on K , and $K_n \subset K_{n+1}$, \tilde{V}_{Q_n} is decreasing on \mathbb{C} . We have, since $\tilde{V}_{Q_n} = Q_n$ q.e. on $\text{supp } \mu_n$,

$$F_n = U^{\mu_n}(z) + Q_n(z) \geq U^{\mu_n}(z) - U^\mu(z), \quad \text{q.e. } z \in \text{supp } \mu_n$$

and $I(\mu_n)$ is finite, hence by the principle of domination (cf., p. 43 of [19]),

$$U^{\mu_n}(z) \leq U^\mu(z) + F_n, \quad z \in \mathbb{C}.$$

Consequently, \tilde{V}_{Q_n} converges in \mathbb{C} to some subharmonic function $f \geq -U^\mu = u$ on \mathbb{C} . Since

$$\forall n \geq 0, \quad u(z) \leq f(z) \leq \tilde{V}_{Q_n}(z) \leq Q_n(z), \quad z \in K_n,$$

Q_n decreases to u on K , and $\cup_n K_n = K \setminus P$ where P is polar, we have that $f = u$ q.e. on K . In particular

$$f(z) \leq u(z), \quad \text{q.e. } z \in K.$$

Again, by the principle of domination (for subharmonic functions),

$$f(z) \leq u(z), \quad z \in \mathbb{C}.$$

Hence $f = u$ on \mathbb{C} , which proves 3.

Since $U^{\mu_n} - F_n$ tends to U^μ pointwise in \mathbb{C} , the fact (i) that F_n tends to 0 simply follows from the behavior of potentials of compactly supported positive measures of total mass r at infinity: each such function decays like $-r \log |z| + o(1/|z|)$. Then fact (ii) that $\mu_n \rightarrow \mu$ weak-* is a consequence of the monotone convergence $U^{\mu_n} - F_n \uparrow U^\mu$ in \mathbb{C} (this would also follow from the stronger convergence in energy (property (iii))).

For the convergence of energies, we observe that

$$I(\mu_n) - rF_n = \int (U^{\mu_n} - F_n) d\mu_n \leq \int U^\mu d\mu_n = \int (U^{\mu_n} - F_n) d\mu + rF_n \leq I(\mu) + rF_n.$$

Hence,

$$\limsup_{n \rightarrow \infty} I(\mu_n) \leq I(\mu).$$

Since we also have that $I(\mu) \leq \liminf_{n \rightarrow \infty} I(\mu_n)$ by the weak-* convergence of μ_n to μ , we obtain that $I(\mu_n)$ tends to $I(\mu)$. \square

Next, we give analogs of Lemmas 5.1 and 5.2 for the vector problem with interaction matrix C .

Lemma 5.3. *Let $\mu = (\mu_1, \dots, \mu_d) \in \mathcal{M}_r(K)$, $K = (K_1, \dots, K_d)$ a tuple of compact sets, $I(\mu_i) < \infty$, $i = 1, \dots, d$. Consider the non-admissible weight $u := (-U^{\mu_1}, \dots, -U^{\mu_d})$ on K . The weighted minimal energy on K is obtained with the measure μ , that is*

$$\forall \nu = (\nu_1, \dots, \nu_d) \in \mathcal{M}_r(K), \quad E_u(\mu) \leq E_u(\nu),$$

with equality if and only if $\nu = \mu$.

Proof. For a tuple of weights Q , we have that

$$E_Q(\nu) - E_Q(\mu) = 2 \sum_{i=1}^d \int (U_i^\mu + Q_i) d(\nu_i - \mu_i) + E(\nu - \mu).$$

Here, with $Q = u$, we simply get

$$E_u(\nu) - E_u(\mu) = E(\nu - \mu),$$

and from [3, Proposition 2.9] (or [18, Chapter 5] if C is positive definite) we know that $E(\nu - \mu)$ is nonnegative and can vanish only when $\nu = \mu$. \square

Given $\mu = (\mu_1, \dots, \mu_d) \in \mathcal{M}_r(K)$ with $I(\mu_i) < \infty$, $i = 1, \dots, d$, we write $u_i := -U^{\mu_i}$ and $u := (-U^{\mu_1}, \dots, -U^{\mu_d})$ as above. By Lemma 5.2, we know that, for each i , there exists a sequence of continuous functions $Q_{n,i}$ defined on K_i and measures $\mu_{n,i}$ on K_i with $\mu_{n,i} \rightarrow \mu_i$ weak-* such that, as $n \rightarrow \infty$,

$$Q_{n,i}(z) \downarrow u_i(z) := -U^{\mu_i}(z), \quad z \in K_i, \quad \text{and} \quad -U^{\mu_{n,i}}(z) + F_{n,i} \downarrow u_i(z), \quad z \in \mathbb{C},$$

where $F_{n,i} \rightarrow 0$ as $n \rightarrow \infty$; $U^{\mu_{n,i}}$ are continuous; and $I(\mu_{n,i}) \rightarrow I(\mu_i)$.

Lemma 5.4. *Given $\mu = (\mu_1, \dots, \mu_d) \in \mathcal{M}_r(K)$ with $I(\mu_i) < \infty$, $i = 1, \dots, d$, let $\mu^{(n)} = (\mu_{n,1}, \dots, \mu_{n,d})$ as above. The following holds true:*

(i) *The tuple of measures $\mu^{(n)}$ tends (component-wise) weak-* to the tuple of measures μ .*

- (ii) The unweighted energy $E(\mu^{(n)})$ tends to $E(\mu)$ as $n \rightarrow \infty$.
(iii) The tuple of weights \widehat{Q}_n such that

$$\widehat{Q}_{n,i}(z) = - \sum_{j=1}^d c_{i,j} U^{\mu_{n,j}}(z), \quad z \in K_i, \quad i = 1, \dots, d,$$

are continuous and the tuple of measures $\mu^{(n)}$ is extremal for the vector problem with interaction matrix C and weight \widehat{Q}_n . Moreover, $E_{\widehat{Q}_n}(\mu^{(n)})$ tends to $E_u(\mu)$ as $n \rightarrow \infty$.

Proof. By Lemma 5.2, each component of $\mu^{(n)}$ tends to the corresponding component of μ . We also know that $I(\mu_{n,i})$ tends to $I(\mu_i)$ for $i = 1, \dots, d$. For the mutual energies, we have

$$\begin{aligned} I(\mu_{n,i}, \mu_{n,j}) - r_j F_{n,i} &= \int (U^{\mu_{n,i}} - F_{n,i}) d\mu_{n,j} \leq \int U^{\mu_i} d\mu_{n,j} \\ &= \int (U^{\mu_{n,j}} - F_{n,j}) d\mu_i + r_i F_{n,j} \leq I(\mu_j, \mu_i) + r_i F_{n,j}. \end{aligned}$$

Using the fact that $F_{n,i} \rightarrow 0$ as $n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} I(\mu_{n,i}, \mu_{n,j}) \leq I(\mu_i, \mu_j).$$

Since the mutual energies are also lowersemicontinuous, we obtain that

$$I(\mu_{n,i}, \mu_{n,j}) \rightarrow I(\mu_i, \mu_j), \quad \text{as } n \rightarrow \infty, \quad (5.1)$$

which shows assertion (ii). For assertion (iii), \widehat{Q}_n is continuous because, for each i , the potential $U^{\mu_{n,i}}$ is continuous. The tuple $\mu^{(n)}$ is extremal for \widehat{Q}_n because the variational inequalities characterizing the solution of the equilibrium problem (see (2.1)-(2.2)) are trivially satisfied. For the convergence of the energies $E_{\widehat{Q}_n}(\mu^{(n)})$ to $E_u(\mu)$, it remains to check that, for all i ,

$$\int \widehat{Q}_{n,i} d\mu_{n,i} \rightarrow - \int U_i^\mu d\mu_i, \quad \text{as } n \rightarrow \infty,$$

which is (5.1) with $i = j$. □

6 The vector energy functionals

In this section we define L^∞ vector energy functionals $\overline{W}, \underline{W}$ and weighted versions $\overline{W}_Q, \underline{W}_Q$, as well as L^2 vector energy functionals $\overline{J}, \underline{J}$ and weighted versions $\overline{J}_Q, \underline{J}_Q$ using (weighted) Bernstein-Markov measures.

We proceed with the definitions. Fix $K = (K_1, \dots, K_d)$, $r_1, \dots, r_d > 0$, an interaction matrix $C \geq 0$, and a strong Bernstein-Markov measure $\nu = (\nu_1, \dots, \nu_d)$; again, as in

Remark 4.13 this Bernstein-Markov property is taken to be with respect to polynomials if all $c_{i,j} \geq 0$ and with respect to rational functions otherwise. Fix a sequence $\{m_k\}$ satisfying (3.1). Given $G \subset \mathcal{M}_r(K)$, for each $k = 1, 2, \dots$ we set

$$\tilde{G}_k := \left\{ \mathbf{a} = (a_{1,1}, \dots, a_{1,m_{1,k}}, a_{2,1}, \dots, a_{d,m_{d,k}}) \in K^k : \left(\frac{r_1}{m_{1,k}} \sum_{j=1}^{m_{1,k}} \delta_{a_{1,j}}, \dots, \frac{r_d}{m_{d,k}} \sum_{j=1}^{m_{d,k}} \delta_{a_{d,j}} \right) \in G \right\} \quad (6.1)$$

and define

$$W_k(G) := \sup\{|VDM_k(\mathbf{a})|^{2|r|^2/|m_k|(|m_k|-1)} : \mathbf{a} \in \tilde{G}_k\}$$

and

$$J_k(G) := \left[\int_{\tilde{G}_k} |VDM_k(\mathbf{a})|^2 d\nu(\mathbf{a}) \right]^{|r|^2/|m_k|(|m_k|-1)}.$$

Definition 6.1. For $\mu \in \mathcal{M}_r(K)$ we define

$$\overline{J}(\mu) := \inf_{G \ni \mu} \overline{J}(G) \text{ where } \overline{J}(G) := \limsup_{k \rightarrow \infty} J_k(G);$$

$$\underline{J}(\mu) := \inf_{G \ni \mu} \underline{J}(G) \text{ where } \underline{J}(G) := \liminf_{k \rightarrow \infty} J_k(G);$$

and

$$\overline{W}(\mu) := \inf_{G \ni \mu} \overline{W}(G) \text{ where } \overline{W}(G) := \limsup_{k \rightarrow \infty} W_k(G);$$

$$\underline{W}(\mu) := \inf_{G \ni \mu} \underline{W}(G) \text{ where } \underline{W}(G) := \liminf_{k \rightarrow \infty} W_k(G).$$

Here the infima are taken over all neighborhoods G of the measure μ in $\mathcal{M}_r(K)$ with the weak-* topology.

Note that $\overline{W}, \underline{W}$ are independent of ν but, a priori, $\overline{J}, \underline{J}$ depend on ν . The weighted versions of these functionals are defined for admissible Q starting with

$$W_k^Q(G) := \sup\{|VDM_k^Q(\mathbf{a})|^{2|r|^2/|m_k|(|m_k|-1)} : \mathbf{a} \in \tilde{G}_k\} \text{ and} \\ J_k^Q(G) := \left[\int_{\tilde{G}_k} |VDM_k^Q(\mathbf{a})|^2 d\nu(\mathbf{a}) \right]^{|r|^2/|m_k|(|m_k|-1)}. \quad (6.2)$$

Definition 6.2. For $\mu \in \mathcal{M}_r(K)$ we define

$$\overline{J}^Q(\mu) := \inf_{G \ni \mu} \overline{J}^Q(G) \text{ where } \overline{J}^Q(G) := \limsup_{k \rightarrow \infty} J_k^Q(G);$$

$$\underline{J}^Q(\mu) := \inf_{G \ni \mu} \underline{J}^Q(G) \text{ where } \underline{J}^Q(G) := \liminf_{k \rightarrow \infty} J_k^Q(G);$$

and

$$\overline{W}^Q(\mu) := \inf_{G \ni \mu} \overline{W}^Q(G) \text{ where } \overline{W}^Q(G) := \limsup_{k \rightarrow \infty} W_k^Q(G);$$

$$\underline{W}^Q(\mu) := \inf_{G \ni \mu} \underline{W}^Q(G) \text{ where } \underline{W}^Q(G) := \liminf_{k \rightarrow \infty} W_k^Q(G).$$

Again the infima are taken over all neighborhoods G of the measure μ in $\mathcal{M}_r(K)$.

The idea behind the $\overline{W}, \underline{W}$ (or $\overline{W}^Q, \underline{W}^Q$) functionals comes from the definition of the (weighted) transfinite diameter in Proposition 3.2. Given μ , we consider all sequences of discrete measures associated to $\mathbf{a} = \mathbf{a}^k \in K^k$ of the form

$$\mu^k := \left(\frac{r_1}{m_{1,k}} \sum_{j=1}^{m_{1,k}} \delta_{a_{1,j}}, \dots, \frac{r_d}{m_{d,k}} \sum_{j=1}^{m_{d,k}} \delta_{a_{d,j}} \right)$$

with $\mu^k \rightarrow \mu$ weak-* and we maximize the asymptotic behavior of the corresponding sequence of numbers $\{|VDM_k(\mathbf{a})|^{2|r|^2/|m_k|(|m_k|-1)}\}$ (or $\{|VDM_k^Q(\mathbf{a})|^{2|r|^2/|m_k|(|m_k|-1)}\}$) over all such $\{\mu^k\}$. The $\overline{J}, \underline{J}$ (or $\overline{J}^Q, \underline{J}^Q$) functionals utilize $L^2(\nu)$ -averages instead. Note that if $\mu^k \rightarrow \mu$ weak-*, then given any neighborhood $G \subset \mathcal{M}_r(K)$ of μ , the tuple of points $\mathbf{a} = \mathbf{a}^k$ belongs to \tilde{G}_k for all k sufficiently large.

All the functionals are uppersemicontinuous on $\mathcal{M}_r(K)$ in the weak-* topology. We write

$$\int_K Q d\mu := \sum_{i=1}^d \int_{K_i} Q_i d\mu_i.$$

Then the following properties hold (and with the $\underline{J}, \underline{J}^Q, \underline{W}, \underline{W}^Q$ functionals as well):

1. $\overline{J}^Q(\mu) \leq \overline{W}^Q(\mu) \leq \delta_Q(K)$ for admissible Q ;
2. $\overline{W}(\mu) = \overline{W}^Q(\mu) \cdot e^{2 \int_K Q d\mu}$ and $\overline{J}(\mu) = \overline{J}^Q(\mu) \cdot e^{2 \int_K Q d\mu}$ for Q continuous.

Proof of 2. First we observe that if $\mu \in \mathcal{M}_r(K)$ and Q is continuous on K , given $\epsilon > 0$, there exists a neighborhood $G \subset \mathcal{M}_r(K)$ of μ with

$$\left| \sum_{i=1}^d \int_{K_i} Q_i \left(d\mu_i - \frac{r_i}{m_{i,k}} \sum_{j=1}^{m_{i,k}} \delta_{a_{i,j}} \right) \right| \leq \epsilon \quad \text{for } \mathbf{a} \in \tilde{G}_k$$

for k sufficiently large. Thus we have

$$-\epsilon - \int_K Q d\mu \leq - \sum_{i=1}^d \frac{r_i}{m_{i,k}} \sum_{j=1}^{m_{i,k}} Q_i(a_{i,j}) \leq \epsilon - \int_K Q d\mu.$$

Recalling (3.2) we get that

$$-\alpha_k(\epsilon + \int_K Q d\mu) \leq - \frac{|r|^2}{|m_k|(|m_k|-1)} \sum_{i=1}^d \frac{m_{i,k}}{r_i} \sum_{j=1}^{m_{i,k}} Q_i(a_{i,j}) \leq \beta_k(\epsilon - \int_K Q d\mu), \quad (6.3)$$

where α_k and β_k tend to 1 as k tends to infinity. Since

$$|VDM_k^Q(\mathbf{a})| := |VDM_k(\mathbf{a})| \cdot \prod_{i=1}^d \prod_{j=1}^{m_{i,k}} e^{-\frac{m_{i,k}}{r_i} Q_i(a_{i,j})},$$

we deduce from (6.3) that

$$|VDM_k(\mathbf{a})|e^{-\alpha_k \frac{|m_k|(|m_k|-1)}{|r|^2}(\epsilon + \int_K Q d\mu)} \leq |VDM_k^Q(\mathbf{a})| \leq |VDM_k(\mathbf{a})|e^{\beta_k \frac{|m_k|(|m_k|-1)}{|r|^2}(\epsilon - \int_K Q d\mu)}.$$

Now we take the supremum over $\mathbf{a} \in \tilde{G}_k$ and take a $|m_k|(|m_k|-1)/2|r|^2$ -th root of each side to get

$$W_k(G)e^{-2\alpha_k(\epsilon + \int_K Q d\mu)} \leq W_k^Q(G) \leq W_k(G)e^{2\beta_k(\epsilon - \int_K Q d\mu)}.$$

Precisely, given $\epsilon > 0$, these inequalities are valid for G a sufficiently small neighborhood of μ . Hence we get, upon taking $\limsup_{k \rightarrow \infty}$, the infimum over $G \ni \mu$, and noting that $\epsilon > 0$ is arbitrary,

$$\overline{W}(\mu) = \overline{W}^Q(\mu) \cdot e^{2 \int_K Q d\mu}$$

as desired. The proof that $\overline{J}(\mu) = \overline{J}^Q(\mu) \cdot e^{2 \int_K Q d\mu}$ is similar. \square

Note from the definition of E and E_Q we have a similar (obvious) relation

$$E_Q(\mu) = E(\mu) + 2 \int_K Q d\mu. \quad (6.4)$$

In particular, $E_Q(\mu^{K,Q}) = E(\mu^{K,Q}) + 2 \int_K Q d\mu^{K,Q}$ so that, using Proposition 3.2,

$$-E(\mu^{K,Q}) = \log \delta_Q(K) + 2 \int_K Q d\mu^{K,Q}. \quad (6.5)$$

Also, from Proposition 3.1,

$$\log \underline{W}(\mu) \leq \log \overline{W}(\mu) \leq -E(\mu). \quad (6.6)$$

We show equality holds in this last relation.

Theorem 6.3. *Let $K = (K_1, \dots, K_d)$ be nonpolar and $Q = (Q_1, \dots, Q_d)$ continuous. Then for any $\mu \in \mathcal{M}_r(K)$,*

$$\log \overline{W}(\mu) = \log \underline{W}(\mu) = -E(\mu) \text{ and} \quad (6.7)$$

$$\log \overline{W}^Q(\mu) = \log \underline{W}^Q(\mu) = -E_Q(\mu). \quad (6.8)$$

Proof. It suffices to prove (6.7) as then (6.8) follows from property 2 and (6.4). We have from (6.6), (6.4), property 2 and Proposition 3.2, for any μ and any Q , the upper bound (inequality) in (6.8) and hence in (6.7):

$$\log \overline{W}^Q(\mu) \leq -E_Q(\mu) \leq -E_Q(\mu^{K,Q}) = \log \delta_Q(K). \quad (6.9)$$

In particular, from 2., for any μ we have

$$\log \overline{W}(\mu) \leq \inf_Q [\log \delta_Q(K) + 2 \int_K Q d\mu].$$

It turns out that equality holds in this last relation (although we will not need/use this).

To get a lower bound on $\log \overline{W}(\mu)$, we begin with the case where $\mu = \mu^{K,v}$ for some $v \in C(K)$. Using Proposition 3.3, if we consider arrays of points $\{\mathbf{Z}_k\} \subset K$ as in (3.3) for which

$$\lim_{k \rightarrow \infty} |VDM_k^v(\mathbf{Z}_k)|^{2|r|^2/|m_k|(|m_k|-1)} = \delta_v(K),$$

we have

$$\left(\frac{r_1}{m_{1,k}} \sum_{j=1}^{m_{1,k}} \delta_{z_{1,j}^{(k)}}, \dots, \frac{r_d}{m_{d,k}} \sum_{j=1}^{m_{d,k}} \delta_{z_{d,j}^{(k)}} \right) \rightarrow \mu^{K,v}$$

weak-*. Thus for any neighborhood G of $\mu^{K,v}$ we have $\delta_v(K) \leq \underline{W}^v(G)$; hence

$$\underline{W}^v(\mu^{K,v}) = \overline{W}^v(\mu^{K,v}) = \delta_v(K). \quad (6.10)$$

Applying 2, (6.10) and (6.5) we obtain (6.7) for $\mu = \mu^{K,v}$:

$$\begin{aligned} \log \underline{W}(\mu^{K,v}) &= \log \overline{W}(\mu^{K,v}) = \log \overline{W}^v(\mu^{K,v}) + 2 \int_K v d\mu^{K,v} \\ &= \log \delta_v(K) + 2 \int_K v d\mu^{K,v} = -E(\mu^{K,v}). \end{aligned} \quad (6.11)$$

Next we take a tuple of measures $\mu \in \mathcal{M}_r(K)$ with $I(\mu_i) < \infty$, $i = 1, \dots, d$. Using Lemma 5.4, for each $i = 1, \dots, d$, there exists a sequence of continuous functions $Q_{n,i}$ defined on K_i and measures $\mu_{n,i}$ such that, as $n \rightarrow \infty$,

$$Q_{n,i}(z) \downarrow u_i(z) := -U^{\mu_i}(z), \quad z \in K_i, \quad \text{and} \quad V_{Q_{n,i}}^*(z) := -U^{\mu_{n,i}}(z) + F_{n,i} \downarrow u_i(z), \quad z \in \mathbb{C}.$$

Here the functions $U^{\mu_{n,i}}$ are continuous; $F_{n,i} \rightarrow 0$; and $I(\mu_{n,i}) \rightarrow I(\mu_i)$. Moreover, writing $\mu^{(n)} = (\mu_{n,1}, \dots, \mu_{n,d})$, from (ii) of the lemma,

$$\lim_{n \rightarrow \infty} E(\mu^{(n)}) = E(\mu), \quad (6.12)$$

and from (iii) of the lemma, for the sequence of continuous functions $\widehat{Q}_n = (\widehat{Q}_{n,1}, \dots, \widehat{Q}_{n,d})$ where $\widehat{Q}_{n,i}(z) = -\sum_{j=1}^d c_{i,j} U^{\mu_{n,j}}(z)$ we have $\mu^{(n)} = \mu^{K, \widehat{Q}_n}$. Thus we can apply the previous case to conclude

$$\log \overline{W}(\mu^{(n)}) = \log \underline{W}(\mu^{(n)}) = -E(\mu^{(n)}).$$

From uppersemicontinuity of the functional $\mu \rightarrow \underline{W}(\mu)$,

$$\limsup_{n \rightarrow \infty} \log \underline{W}(\mu^{(n)}) = \limsup_{n \rightarrow \infty} \log \overline{W}(\mu^{(n)}) = \limsup_{n \rightarrow \infty} [-E(\mu^{(n)})] \leq \log \underline{W}(\mu).$$

But from (6.12) we see that the limit exists and

$$\lim_{n \rightarrow \infty} \log \underline{W}(\mu^{(n)}) = \lim_{n \rightarrow \infty} [-E(\mu^{(n)})] = -E(\mu) \leq \log \underline{W}(\mu).$$

Together with (6.6) we have

$$\log \underline{W}(\mu) = \log \overline{W}(\mu) = -E(\mu).$$

To finish the proof, we must show that if $\mu \in \mathcal{M}_r(K)$ satisfies $I(\mu_i) = \infty$ for some $i = 1, \dots, d$, then $E(\mu) = \infty$ and $\overline{W}(\mu) = 0$. The fact that $E(\mu) = \infty$ is clear; then the upper bound in (6.6) shows that $\overline{W}(\mu) = 0$. \square

We now consider the $\overline{J}, \underline{J}, J^Q$ and \underline{J}^Q functionals.

Theorem 6.4. *Let $K = (K_1, \dots, K_d)$ be nonpolar and $Q = (Q_1, \dots, Q_d)$ continuous and let $\nu \in \mathcal{M}_1(K)$ satisfy a strong Bernstein-Markov property. Then for any $\mu \in \mathcal{M}_r(K)$,*

$$\log \overline{J}(\mu) = \log \overline{W}(\mu) = \log \underline{J}(\mu) = \log \underline{W}(\mu) = -E(\mu) \quad (6.13)$$

and

$$\log \overline{J}^Q(\mu) = \log \overline{W}^Q(\mu) = \log \underline{J}^Q(\mu) = \log \underline{W}^Q(\mu) = -E_Q(\mu). \quad (6.14)$$

Proof. As in the previous proof, it suffices to show (6.13) since (6.14) follows from property 2. We have the upper bound as before; for the lower bound, we consider the case where $\mu = \mu^{K,v}$ for $v \in C(K)$. We show the analogue of (6.11) for $\overline{J}, \underline{J}$:

$$\log \overline{J}(\mu^{K,v}) = \log \underline{J}(\mu^{K,v}) = \log \delta_v(K) + 2 \int_K v d\mu^{K,v}. \quad (6.15)$$

Then (6.15) will imply that

$$\log \overline{J}(\mu^{K,v}) = \log \overline{W}(\mu^{K,v}) = \log \underline{J}(\mu^{K,v}) = \log \underline{W}(\mu^{K,v}) = -E(\mu^{K,v})$$

and hence

$$\log \overline{J}(\mu) = \log \overline{W}(\mu) = \log \underline{J}(\mu) = \log \underline{W}(\mu) = -E(\mu)$$

for arbitrary $\mu \in \mathcal{M}_r(K)$ following the proof of Theorem 6.3. This proves (6.13). To prove (6.15), we first verify the following.

Claim: Fix a neighborhood G of $\mu^{K,v}$. For $\eta > 0$, define $A_{k,\eta}$ as in (4.5) with $Q = v$. Given a sequence $\{\eta_j\}$ with $\eta_j \downarrow 0$, there exists a j_0 and a k_0 such that

$$\forall j \geq j_0, \quad \forall k \geq k_0, \quad A_{k,\eta_j} \subset \tilde{G}_k. \quad (6.16)$$

We prove (6.16) by contradiction: if false, there are sequences $\{k_l\}$ and $\{j_l\}$ tending to infinity such that for all l sufficiently large we can find a point $\mathbf{Z}_{\mathbf{k}_l} \in A_{k_l,\eta_{j_l}} \setminus \tilde{G}_{k_l}$. But

$$\mu^l := \left((r_1/m_{1,k_l}) \sum_{i=1}^{m_{1,k_l}} \delta_{z_{1,i}}, \dots, (r_d/m_{d,k_l}) \sum_{i=1}^{m_{d,k_l}} \delta_{z_{d,i}} \right) \notin G$$

for l sufficiently large contradicts Proposition 3.3 since $\mathbf{Z}_{\mathbf{k}_l} \in A_{k_l, \eta_{j_l}}$ and $\eta_{j_l} \rightarrow 0$ imply $\mu^l \rightarrow \mu^{K,v}$ weak-*. This proves the claim.

Fix a neighborhood G of $\mu^{K,v}$ and a sequence $\{\eta_j\}$ with $\eta_j \downarrow 0$. For $j \geq j_0$, choose $k = k_j$ large enough so that the inclusion in (6.16) holds true as well as

$$Prob_{k_j}(K^{k_j} \setminus A_{k_j, \eta_j}) \leq \left(1 - \frac{\eta_j}{2\delta_v(K)}\right)^{|m_{k_j}|(|m_{k_j}|-1)/|r|^2} \nu(K^{k_j}), \quad (6.17)$$

and

$$\left(1 - \frac{\eta_j}{2\delta_v(K)}\right)^{|m_{k_j}|(|m_{k_j}|-1)/|r|^2} \nu(K^{k_j}) \rightarrow 0 \quad \text{as } j \rightarrow \infty, \quad (6.18)$$

which is possible (for (6.17) we make use of Corollary 4.15). In view of (6.16), (4.4) and (6.17), we have

$$\begin{aligned} \frac{1}{Z_{k_j}^v} \int_{\tilde{G}_{k_j}} |VDM_{k_j}^v(\mathbf{Z}_{\mathbf{k}_j})|^2 d\nu(\mathbf{Z}_{\mathbf{k}_j}) &\geq \frac{1}{Z_{k_j}^v} \int_{A_{k_j, \eta_j}} |VDM_{k_j}^v(\mathbf{Z}_{\mathbf{k}_j})|^2 d\nu(\mathbf{Z}_{\mathbf{k}_j}) \\ &\geq 1 - \left(1 - \frac{\eta_j}{2\delta_v(K)}\right)^{|m_{k_j}|(|m_{k_j}|-1)/|r|^2} \nu(K^{k_j}). \end{aligned} \quad (6.19)$$

Note that, because of (6.18), the lower bound in (6.19) tends to 1 as $j \rightarrow \infty$. Then, since ν satisfies a strong Bernstein-Markov property, we derive, along with Proposition 4.14, that

$$\liminf_{j \rightarrow \infty} \frac{|r|^2}{|m_{k_j}|(|m_{k_j}|-1)} \log \int_{\tilde{G}_{k_j}} |VDM_{k_j}^v(\mathbf{Z}_{\mathbf{k}_j})|^2 d\nu(\mathbf{Z}_{\mathbf{k}_j}) \geq \log \delta_v(K).$$

Giving any sequence of positive integers $\{k\}$ we can find a subsequence $\{k_j\}$ as above corresponding to some $\eta_j \downarrow 0$; hence

$$\liminf_{k \rightarrow \infty} \frac{|r|^2}{|m_k|(|m_k|-1)} \log \int_{\tilde{G}_k} |VDM_k^v(\mathbf{Z}_k)|^2 d\nu(\mathbf{Z}_k) \geq \log \delta_v(K).$$

It follows that

$$\log \underline{J}^v(G) \geq \log \delta_v(K).$$

Taking the infimum over all neighborhoods G of $\mu^{K,v}$ we obtain

$$\log \underline{J}^v(\mu^{K,v}) \geq \log \delta_v(K).$$

Thus we have the version of (6.10) with \overline{J}^v and \underline{J}^v :

$$\log \underline{J}^v(\mu^{K,v}) = \log \overline{J}^v(\mu^{K,v}) = \log \delta_v(K). \quad (6.20)$$

Using 2. with $\mu = \mu^{K,v}$, from (6.20) we obtain (6.15). \square

Remark 6.5. The equality of \overline{J}^Q and \underline{J}^Q is the basis for the proof of our large deviation principle in the next section. From now on, we simply use the notation J, J^Q, W, W^Q without the overline or underline. Note that, in particular, these functionals are independent of the sequence $\{m_k\}$ satisfying (3.1); and J, J^Q are independent of the strong Bernstein-Markov measure ν .

7 Large deviation principle

In this section, $K = (K_1, \dots, K_d)$ are nonpolar disjoint compact sets in \mathbb{C} . We fix an interaction matrix $C \geq 0$, positive numbers r_1, \dots, r_d , as well as a measure $\nu = (\nu_1, \dots, \nu_d)$ satisfying a strong Bernstein-Markov property and a tuple of continuous weights (Q_1, \dots, Q_d) . Again, this Bernstein-Markov property is taken to be with respect to polynomials if all $c_{i,j} \geq 0$ and with respect to rational functions otherwise. We take a sequence of tuples of positive integers $\{m_k\}$ satisfying (3.1). As before, we associate to a set of points

$$\mathbf{Z}_k := (z_{1,1}, \dots, z_{1,m_{1,k}}, z_{2,1}, \dots, z_{d,m_{d,k}}) \in K_1^{m_{1,k}} \times \dots \times K_d^{m_{d,k}} = K^k$$

the measure

$$\mu^k := \left(\frac{r_1}{m_{1,k}} \sum_{j=1}^{m_{1,k}} \delta_{z_{1,j}^{(k)}}, \dots, \frac{r_d}{m_{d,k}} \sum_{j=1}^{m_{d,k}} \delta_{z_{d,j}^{(k)}} \right) \in \mathcal{M}_r(K).$$

Define $j_k : K^k \rightarrow \mathcal{M}_r(K)$ via

$$j_k(\mathbf{Z}_k) = \mu^k.$$

From (4.4), $\sigma_k := (j_k)_*(\text{Prob}_k)$ is a probability measure on $\mathcal{M}_r(K)$. We can be more precise about this definition. For a Borel set $G \subset \mathcal{M}_r(K)$,

$$\sigma_k(G) = \frac{1}{Z_k^Q} \int_{\tilde{G}_k} |VDM_k^Q(\mathbf{Z}_k)|^2 d\nu(\mathbf{Z}_k) \quad (7.1)$$

where \tilde{G}_k is defined in (6.1).

Theorem 7.1. *The sequence $\{\sigma_k = (j_k)_*(\text{Prob}_k)\}$ of probability measures on $\mathcal{M}_r(K)$ satisfies a **large deviation principle (LDP)** with speed $|m_k|(|m_k| - 1)/2|r|^2$ and good rate function $\mathcal{I} := \mathcal{I}_{K,Q}$ where*

$$\mathcal{I}(\mu) := \log J^Q(\mu^{K,Q}) - \log J^Q(\mu) = \log W^Q(\mu^{K,Q}) - \log W^Q(\mu) = E_Q(\mu) - E_Q(\mu^{K,Q}). \quad (7.2)$$

Remark 7.2. For basic notions involving LDP, we refer the reader to [11]. Note that for *each* sequence of tuples of positive integers $\{m_k\}$ satisfying (3.1) and *each* strong Bernstein-Markov measure ν we get an LDP where the speed depends on m_k but the rate function is independent of both m_k and ν .

The following is a special case of a basic general existence result for a LDP given in Theorem 4.1.11 in [11].

Proposition 7.3. *Let $\{\sigma_\epsilon\}$ be a family of probability measures on $\mathcal{M}_r(K)$. Let \mathcal{B} be a base for the topology of $\mathcal{M}_r(K)$. For $\mu \in \mathcal{M}_r(K)$ let*

$$\mathcal{I}(\mu) := - \inf_{\{G \in \mathcal{B} : \mu \in G\}} \left(\liminf_{\epsilon \rightarrow 0} \epsilon \log \sigma_\epsilon(G) \right).$$

Suppose for all $\mu \in \mathcal{M}_r(K)$,

$$\mathcal{I}(\mu) = - \inf_{\{G \in \mathcal{B} : \mu \in G\}} \left(\limsup_{\epsilon \rightarrow 0} \epsilon \log \sigma_\epsilon(G) \right).$$

Then $\{\sigma_\epsilon\}$ satisfies a LDP with rate function $\mathcal{I}(\mu)$ and speed $1/\epsilon$.

Proof. (of Theorem 7.1): As a base \mathcal{B} for the topology of $\mathcal{M}_r(K)$, we can, e.g., take all open sets. For $\{\sigma_\epsilon\}$, we take the sequence of probability measures $\{\sigma_k\}$ on $\mathcal{M}_r(K)$ and we take $\epsilon = 2|r|^2/|m_k|(|m_k| - 1)$. For $G \in \mathcal{B}$

$$\frac{2|r|^2}{|m_k|(|m_k| - 1)} \log \sigma_k(G) = \log J_k^Q(G) - \frac{2|r|^2}{|m_k|(|m_k| - 1)} \log Z_k^Q$$

using (6.2) and (7.1). From Proposition 4.14, and (6.20) with Q ,

$$\lim_{k \rightarrow \infty} \frac{2|r|^2}{|m_k|(|m_k| - 1)} \log Z_k^Q = \log \delta_Q(K) = \log J^Q(\mu^{K,Q});$$

and by Theorem 6.4, if $E(\mu) < \infty$,

$$\inf_{G \ni \mu} \limsup_{k \rightarrow \infty} \log J_k^Q(G) = \inf_{G \ni \mu} \liminf_{k \rightarrow \infty} \log J_k^Q(G) = \log J^Q(\mu).$$

If, on the other hand, $E(\mu) = \infty$, then $J(\mu) = W(\mu) = 0$ and hence for $G \ni \mu$

$$\lim_{k \rightarrow \infty} \log J_k^Q(G) = -\infty.$$

Thus by Proposition 7.3, $\{\sigma_k\}$ satisfies an LDP with rate function

$$\mathcal{I}(\mu) := \log J^Q(\mu^{K,Q}) - \log J^Q(\mu) = E_Q(\mu) - E_Q(\mu^{K,Q})$$

and speed $|m_k|(|m_k| - 1)/2|r|^2$. This rate function is good since $\mathcal{M}_r(K)$ is compact. \square

8 Possibly intersecting sets

Many of the results in the paper remain valid for nonpolar compact sets K_1, \dots, K_d that are not necessarily disjoint. We make the standing assumption, as in [3], that

- (i) *There exists a vector (y_1, \dots, y_d) in the range of C such that if $K_i \cap K_j \neq \emptyset$, then $y_i y_j > 0$.*
- (ii) *If $\{i_1, \dots, i_m\} \subset \{1, 2, \dots, d\}$ are indices such that the m columns $\{C_{i_j}\}_{j=1, \dots, m}$ of C are linearly dependent, then $\text{cap}(\cap_{j=1}^m K_{i_j}) = 0$.*

These assumptions are automatically satisfied if C is positive definite. In section 2, the existence and uniqueness of a minimizing d -tuple of measures for the energy E over $\mu \in \mathcal{M}_r(K)$ and in the weighted case for the energy E_Q is covered in Theorem 1.8 of [3]. Indeed, it is proved that utilizing the partial potentials

$$U_i^\mu = \sum_{j=1}^d c_{i,j} U^{\mu_j}, \quad i = 1, \dots, d,$$

a measure μ minimizes the weighted energy E_Q if and only if there exist constants F_1, \dots, F_d such that the variational inequalities

$$\begin{aligned} U_i^\mu(z) + Q_i &\geq F_i, & \text{q.e. } z \in K_i, & \quad i = 1, \dots, d, \\ U_i^\mu(z) + Q_i &\leq F_i, & \mu_i\text{-a.e. } z \in K_i, & \quad i = 1, \dots, d \end{aligned}$$

hold.

We claim that if, in addition, we assume that

$$c_{i,j} \text{ is nonnegative if } K_i \cap K_j \neq \emptyset \quad (8.1)$$

then all of the results in sections 3-7 remain true. In particular, the Angelesco ensembles satisfying (8.1) are covered in this setting as are the Nikishin ensembles when the sets K_i and $K_{i\pm 1}$ are disjoint. We indicate the minor modifications of the proofs/results needed in these sections with the above hypotheses.

The equality (3.10) will now be replaced by an inequality with \liminf :

$$I(\mu_i, \mu_j) \leq \liminf_{k \rightarrow \infty} I(\mu_i^k, \mu_j^k) = \liminf_{k \rightarrow \infty} \frac{r_i r_j}{m_{i,k} m_{j,k}} \sum_{l=1}^{m_{i,k}} \sum_{p=1}^{m_{j,k}} \log \frac{1}{|z_{i,l}^{(k)} - z_{j,p}^{(k)}|}.$$

This leaves the rest of the proof of Proposition 3.1, and the results in section 3, unchanged. Note that the scaling result, (3.12), still holds.

Hypothesis (8.1) obviates the need for any modifications of the (vector) Bernstein-Markov properties in section 4. The result from [3, Proposition 2.9] used in Lemma 5.3 that $E(\nu - \mu)$ is nonnegative and can vanish only when $\nu = \mu$ remains true; and since the variational inequalities listed above characterizing the solution of the equilibrium problem remain valid, all of the arguments in section 5 are unaltered.

The results in sections 6 and 7 rest solely on the preliminaries in the previous sections; thus, *Theorems 6.3, 6.4, and the LDP Theorem 7.1, remain true for nonpolar compact sets K_1, \dots, K_d that are not necessarily disjoint provided assumptions (i), (ii) and (8.1) are satisfied.*

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